Effective Resistance in Simplicial Complexes: Generalizations and Properties

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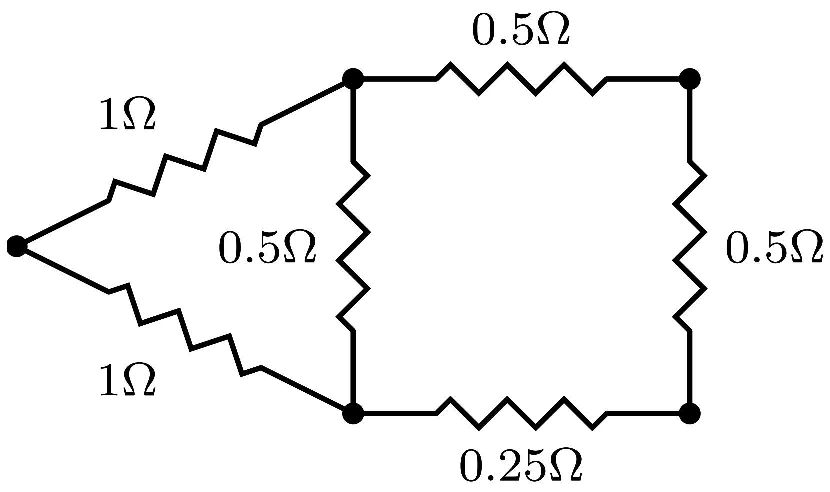
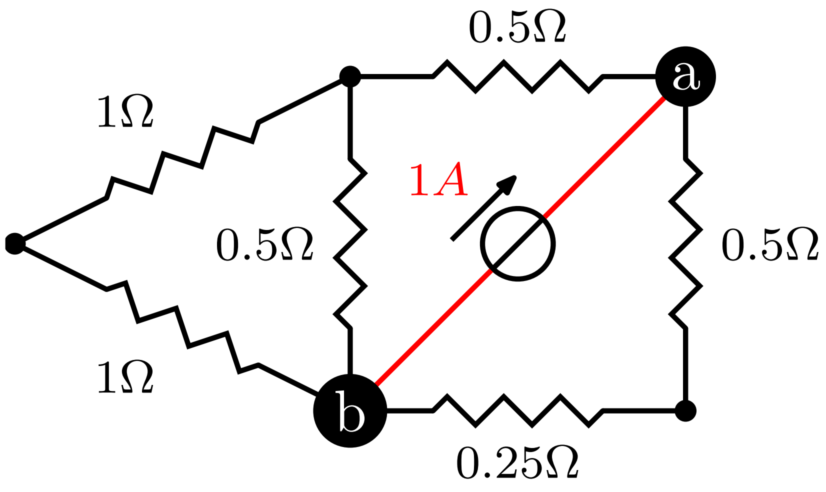
Seminario de TDA - CUNEF

From circuits to graphs



Electric network of resistors

From circuits to graphs



Electric network of resistors

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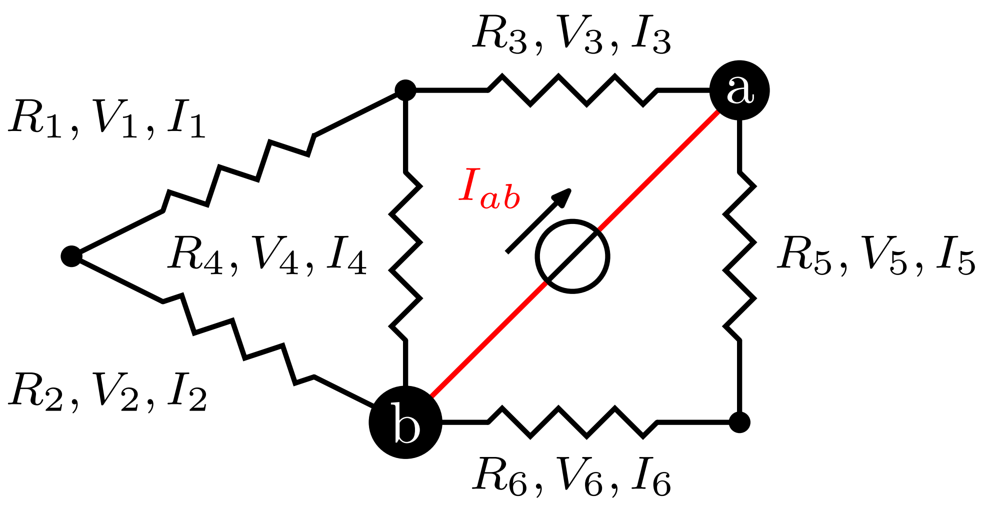
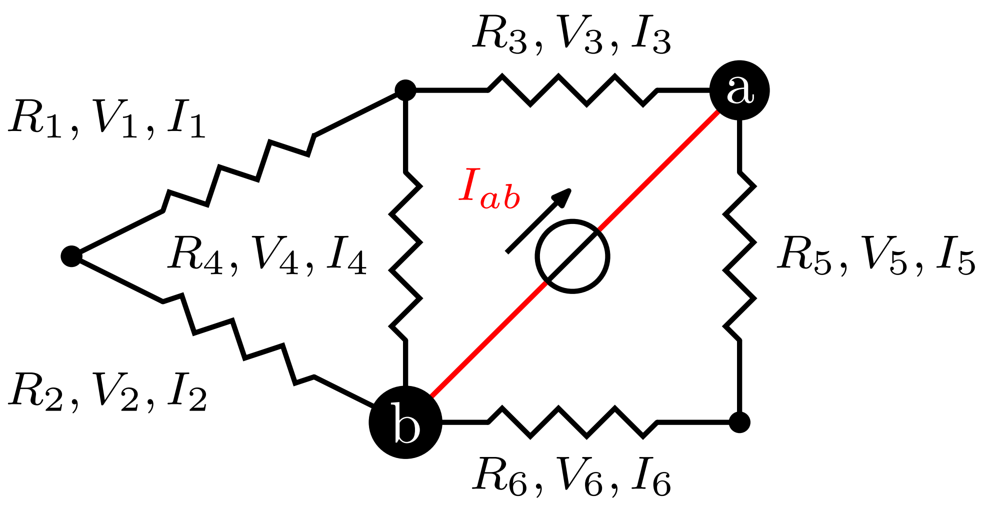


Electric network of resistors

From circuits to graphs

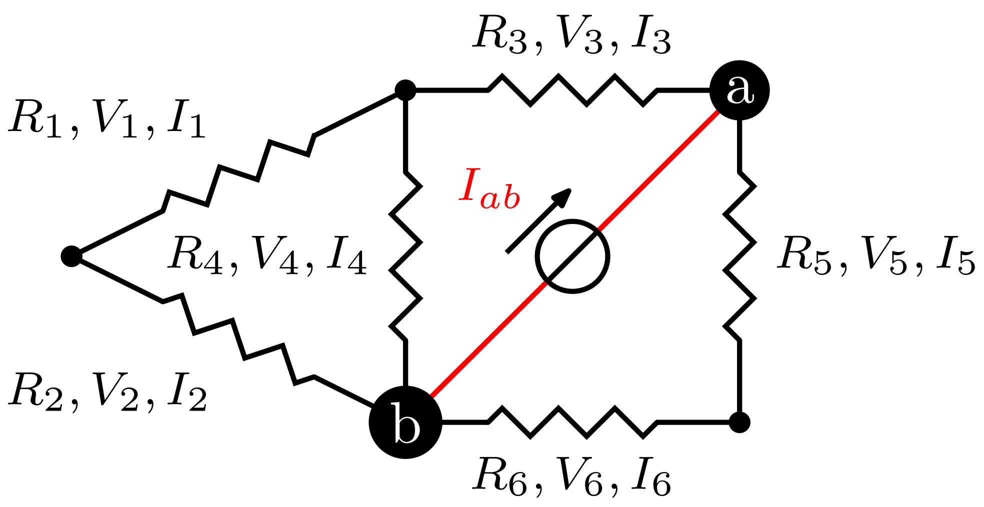


Electric network of resistors

1. **Kirchhoff's voltage law:** the sum or all voltages around a loop is zero
2. **Kirchhoff's current law:** the current injected at each vertex equals the current extracted from that vertex
3. **Ohm's law:** the current at each point is proportional to the voltage, the proportionality constant being the *resistance*

$$V = IR$$

From circuits to graphs



Electric network of resistors

Effective resistance

For *a* and *b* vertices, the effective resistance is the *total resistance* when a current I_{ab} is injected into *a* and extracted from *b* (usually, simply set $I_{ab} = 1$)

$$R_{ab} := \frac{v_a - v_b}{I_{ab}}$$

Thomson Principle

Among all the possible current flows in the circuit, the energy dissipated by the flow that follows the electrical laws is least.

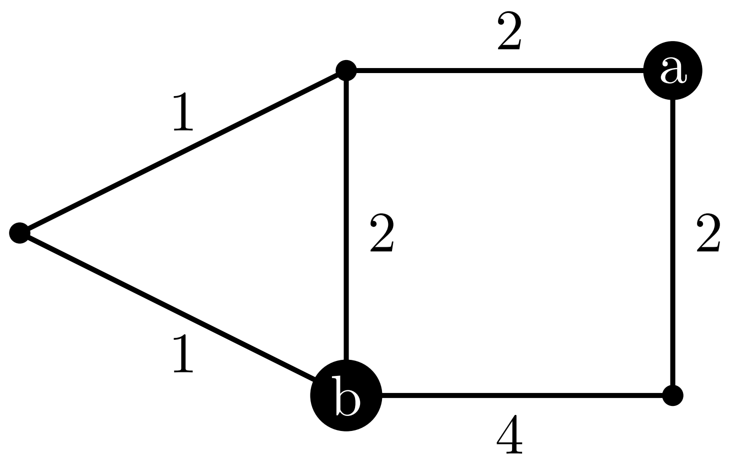
$$E = I_{ab}^2 R_{ab}$$

From circuits to graphs

Weighted graph

Conductance

$$G = (V, E, w) \qquad w : E \to \mathbb{R}_{>0}, \quad w(e) = R_e^{-1}$$



Weight matrix

$$W = \begin{pmatrix} w(e_1) & 0 & \dots & 0 \\ 0 & w(e_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots w(e_m) & \end{pmatrix} \in \mathbb{R}^{|E| \times |E|}$$

Incidence matrix

$$B \in \mathbb{R}^{|V| \times |E|}, \quad B(v,e) = \begin{cases} 1 & \text{if } v = \text{head of } e \\ -1 & \text{if } v = \text{tail of } e \\ 0 & \text{otherwise} \end{cases}$$

Graph Laplacian matrix

$$L = B\,W\,B^\top$$

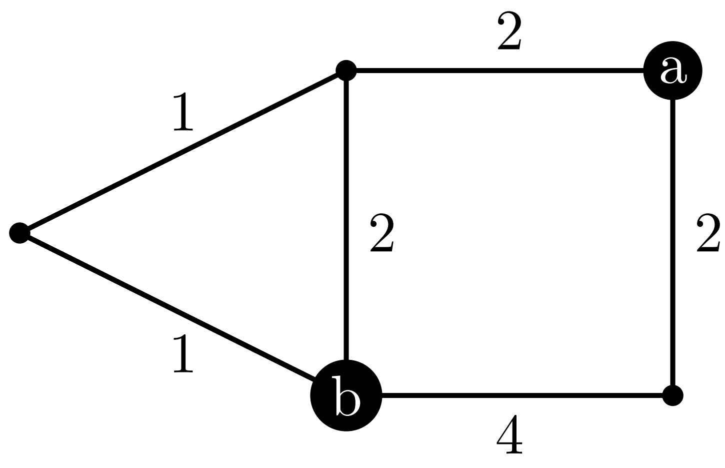
From circuits to graphs

Weighted graph

Conductance

$$G = (V, E, w) \qquad w : E \to \mathbb{R}_{>0}, \quad w(e) = R_e^{-1}$$

Inject I_{ab} current in *a* and extract from *b*



Potential vector

$\phi = (\phi_{v_1}, \dots, \phi_{v_n})$

Current vector

$I = (i_{e_1}, \dots, i_{e_m})$

Resistance vector

$R = (R_{e_1}, \dots, R_{e_m})$

Electric laws

$$L\phi = I_{ab}(e_a - e_b)$$

Potential vector $\phi = (\phi_{v_1}, \dots, \phi_{v_n})$

Vertex indicator vector

KCL:

$$\sum_{v \in V} i_{uv} = \begin{cases} I_{ab} & \text{if } u = a, \\ -I_{ab} & \text{if } u = b, \\ 0 & \text{otherwise.} \end{cases}$$

u-th coordinate of $I_{ab}(e_a - e_b)$

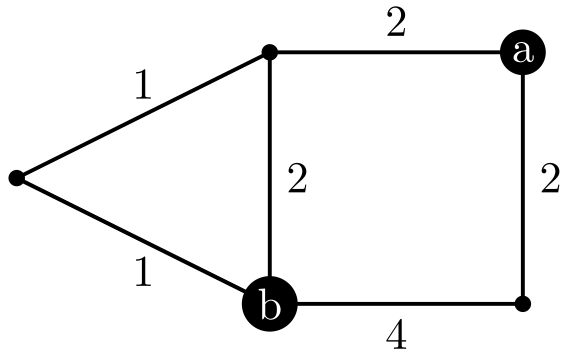
OL:

$$\sum_{v \in V} i_{uv} = \sum_{v \in V} \frac{\phi_u - \phi_v}{R_{uv}} = \phi_u \sum_{v \in V} w_{uv} - \sum_{v \in V} \phi_v w_{uv}$$

u-th coordinate of $L\phi$

Effective resistance in graphs

Definition



Effective resistance

$$\tilde{r}_{ab} = (e_a - e_b)^\top L^+ (e_a - e_b)$$

Pseudo-inverse matrix

Idea of the proof:

$$\begin{aligned}
\tilde{r}_{ab} &= \frac{\phi_a - \phi_b}{I_{ab}} && \text{(definition)} \\
&= \frac{(e_a - e_b)^\top \phi}{I_{ab}} && (\phi = (\phi_{v_1}, \ldots, \phi_{v_n})) \\
&= (e_a - e_b)^\top L^+ (e_a - e_b) && \text{(Kirchhoff's laws)}
\end{aligned}$$

Vertex-based ER matrix

$$\tilde{R} \in \mathbb{R}^{|V| \times |V|}, \quad \tilde{R}(i,j) = \tilde{r}_{v_i, v_j}$$

Edge-based ER matrix

$$R \in \mathbb{R}^{|E| \times |E|}, \quad R = B^\top L^+ B$$

$$\text{if } e = (a,b) \implies R(e,e) = \tilde{r}_{ab} = r_e$$

Effective resistance in graphs

Properties

$$\tilde{r}_{ab} = (e_a - e_b)^\top L^+ (e_a - e_b)$$
$$\text{if } e = (a,b) \implies R(e,e) = \tilde{r}_{ab} = r_e$$

- **Kirchhoff's theorem** (1847): the effective resistance is proportional to the probability of appearance of an edge in a *random spanning tree T* in *G*

$$\Pr[e \in T] = w(e) \cdot r_e$$

- **Foster's theorem** (1949): $$\sum_{e \in E} w(e) \cdot r_e = |V| - \beta_0(G)$$

Number of connected components

- It is **proportional** to the **commute time** (Chandra et al, 1996): $$C_{ab} = 2\,|E|\,\tilde{r}_{ab}$$

- It defines a **metric** between the nodes in a graph (cf. Devriendt 2022)

- Applications as a **centrality measure** (Newman, 2005), in **sparsification** (Spielman and Srivastava, 2011), to address **over-squashing** (Black et al., 2023)…

Effective resistance in simplicial complexes

- **Kook and Lee** (2018): extend to *simplicial networks*, prove Foster's theorem and establish enumeration theorems
- **Osting et al.,** (2020): define an ER matrix to extend *sparsification* techniques
- **Black and Maxwell** (2021): define ER for cycles, as a minimizer of the *flow of energy*

…but all these approaches are, at first sight, very different from each other!

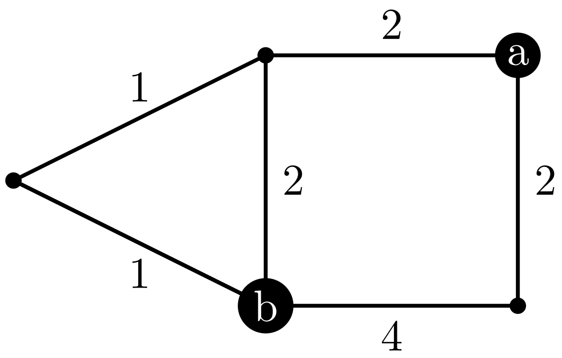
This talk

- Introduce a **basis-free bilinear form** for effective resistance in simplicial complexes
- **Unify** all existing **matrix definitions**
- Define a **metric** on p-cycles
- Generalize **Foster's theorem**
- Use it to count **higher-dimensional spanning trees**

Circuit theory in algebraic topology terms

(Baez, 2018)

$G = (V, E, w) \qquad w : E \to \mathbb{R}_{>0}, \quad w(e) = R_e^{-1}$



Setup

- Voltage: $f \in C^1(G)$ (1-cochain)
- Current: $\alpha \in C_1(G)$ (1-chain)
- Terminals: $V_{\text{in}} \cup V_{\text{out}}$

$$C^0(G) \xrightarrow{\delta_0} C^1(G)$$
$$C_0(G) \xleftarrow{\partial_1} C_1(G)$$

Potential

Boundary current

Kirchhoff's voltage law
$\exists \phi \in C^0(G) \quad \text{s.t.} \quad f = \delta_0 \phi$

- A *physically valid* voltage is a coboundary
- The voltage vanishes on 1-cycles

$$c \in \ker \partial_1 \implies f(c) = \delta_0 \phi(c) = \phi(\partial_1 c) = 0$$

Kirchhoff's current law
$\exists \beta \in C_0(V_{\text{in}} \cup V_{\text{out}}) \quad \text{s.t.} \quad \partial_1 \alpha = \beta$

- A *physically valid* current has as boundary the *terminals*
- Net current vanishes at interior vertices and is non-zero at terminals

Inner products and Ohm's law

(Baez, 2018)

$$G = (V, E, w) \qquad w : E \to \mathbb{R}_{>0}, \quad w(e) = R_e^{-1}$$

Inner products

Chain space

$$\langle e, e' \rangle_{C_1} := \begin{cases} w(e)^{-1} = R_e & \text{if } e = e', \\ 0 & \text{otherwise} \end{cases}$$

Cochain space

$$\langle e^\vee, e'^\vee \rangle_{C^1} := \langle e, e' \rangle_{C_1}^{-1}$$

Define adjoints

Given $f : U \to V$, its *adjoint* is $f^* : V \to U$ such that

$$\langle u, f^*(v) \rangle_U = \langle f(u), v \rangle_V, \quad \forall u \in U, \; v \in V$$

Define musical isomorphisms

$$\begin{array}{ccc} C^0(G) & \xleftarrow{\delta_0^*} & C^1(G) \\ \flat_0 \uparrow & & \uparrow \flat_1 \\ C_0(G) & \xleftarrow{\partial_1} & C_1(G) \end{array}$$

$$\flat_1(\alpha) := \langle \alpha, \bullet \rangle_{C_1}$$

Ohm's law
$f = \flat_1 \alpha$

- Given *f* or α, the other is uniquely determined
- This extends the physical law: for an edge *e* we have

$$f(e) = \flat_1 \alpha(e) = \langle \alpha, e \rangle_{C_1} = R_e \; \alpha_e$$

Generalization to simplicial complexes

Proposition (existence of currents, potentials and voltages in all dimensions)

Given $\beta \in C_{p-1}(K)$, the system of equations

$$\begin{cases} \delta_{p-1}\phi = f & \text{(KVL)} \\ \partial_p \alpha = \beta & \text{(KCL)} \\ \flat_p \alpha = f & \text{(OL)} \end{cases}$$

in the unknowns $\alpha \in C_p(K)$, $f \in C^p(K)$, and $\phi \in C^{p-1}(K)$ admits solutions if and only if the equation

$$\boxed{\mathcal{L}_{up}^{p-1}\phi = \flat_{p-1}\beta}$$

in the unknown ϕ does, with $\mathcal{L}_{up}^{p-1} := \delta_{p-1}^{*}\delta_{p-1}$. For this, it is sufficient that β is a boundary.

The proof is just a matter of chasing the diagram:

$$\begin{array}{ccccccc} \ldots & \longleftarrow & C^{p-1}(K) & \xleftarrow{\delta_{p-1}^{*}} & C^{p}(K) & \longleftarrow & \ldots \\ & & \uparrow{\scriptstyle \flat_{p-1}} & & \uparrow{\scriptstyle \flat_{p}} & & \\ \ldots & \longleftarrow & C_{p-1}(K) & \xleftarrow{\partial_p} & C_{p}(K) & \longleftarrow & \ldots \end{array}$$

Effective resistance bilinear form

If we inject a unit of current in a and extract it from b, the boundary current is $\beta = a - b \in C_0(K)$. The difference of potential induced by this is

$$\langle \phi, \flat_0 \beta \rangle_{C^0} = \langle (\mathcal{L}^0_{\text{up}})^+ \flat_0 \beta, \flat_0 \beta \rangle_{C^0}$$

$\phi = (\mathcal{L}^0_{\text{up}})^+ \flat_0 \beta$ is a potential

$B_p(K) = \operatorname{im} \partial_{p+1}$ $\quad$ $\mathcal{L}^{\text{up}}_p = \partial_{p+1} \partial^*_{p+1}$

ER bilinear form in p-boundaries

$$\tilde{\mathcal{R}}_p : B_p(K) \times B_p(K) \to \mathbb{R}$$
$$(\beta, \beta') \mapsto \tilde{\mathcal{R}}_p(\beta, \beta') := \left\langle (\mathcal{L}^{\text{up}}_p)^+ \beta, \beta' \right\rangle_{C_p}$$

ER bilinear form in p-chains

$$\mathcal{R}_p : C_p(K) \times C_p(K) \to \mathbb{R}$$
$$(\alpha, \alpha') \mapsto \mathcal{R}_p(\alpha, \alpha') := \left\langle \partial^*_p (\mathcal{L}^{\text{up}}_{p-1})^+ \partial_p \alpha, \alpha' \right\rangle_{C_p}$$

For $p = 0$, if $\beta = \beta' = a - b$, this recovers the effective resistance $\tilde{r}_{ab} = \langle (\mathcal{L}^{\text{up}}_0)^+ \beta, \beta' \rangle_{C_0}$.

For $\beta = \partial_p \alpha$ and $\beta' = \partial_p \alpha'$, we have $\tilde{\mathcal{R}}_{p-1}(\beta, \beta') = \mathcal{R}(\alpha, \alpha')$

Effective resistance of chains, effective resistance operator and matrix form

ER of a chain

For $\alpha \in C_p(K)$, we have $r_\alpha = \mathcal{R}_p(\alpha, \alpha)$

ER operator

$$\mathcal{T}_p := \partial_p^*(\mathcal{L}_{p-1}^{\mathrm{up}})^+ \partial_p$$

$\mathcal{T}_p : C_p(K) \to C_p(K)$ is an orthogonal projection onto $\operatorname{im} \partial_p^*$

$\mathcal{T}_p$ is self-adjoint and semi-positive definite, so
$\mathcal{R}_p$ is symmetric and semi-positive definite.

Matrix form

- Taking the *standard basis* in $C_p(K)$: $\mathcal{B}_p = \{\sigma_1, \ldots, \sigma_{|K_p|}\}$
- Considering *unit weights* in $C_{p-1}(K)$

$$[\mathcal{R}]_{\mathcal{B}_p, \mathcal{B}_p} = B_p^\top (B_p W_p B_p^\top)^+ B_p$$

- Recovers graph definition for *p=1*
- Coincides with the definition for simplicial complexes by Osting et al., (2020)
- Similarly for Kook and Lee (2018) and Black and Maxwell (2021)

Metric properties

Theorem

- The following is a *pseudometric* on $C_p(K)$

$$d_p(\alpha, \alpha') \coloneqq \sqrt{\mathcal{R}_p(\alpha - \alpha', \alpha - \alpha')} = \left\|\mathcal{T}_p(\alpha - \alpha')\right\|_{C_p}$$

- If $\tilde{H}_p(K) = 0$, the following is a *metric* on $Z_p(K) = \ker \tilde{\partial}_p$

$$\tilde{d}_p(c, c') \coloneqq \sqrt{\mathcal{R}_{p+1}(\beta, \beta)} = \|\mathcal{T}_{p+1}(\beta)\|_{C_{p+1}}$$

where $\partial_{p+1}\beta = c - c'$

Well-defined as $\mathcal{T}_{p+1}(\beta) = \partial^*_{p+1}(\mathcal{L}^{\mathrm{up}}_p)\partial_{p+1}(\beta)$ only depends on $\partial_{p+1}(\beta) = c - c'$

For a connected graph, $\tilde{H}_0(K) = 0$ and $\tilde{d}_0(a, b)$ recovers the *effective resistance distance.*

Foster's theorem

Higher dimensional Foster's theorem

If $\{\lambda_i(\mathcal{T}_p) : i \in I\}$ are the *eigenvalues* of $\mathcal{T}_p$, then

$$\sum_{i \in I} \lambda_i(\mathcal{T}_p) = |K_{p-1}| - \mathrm{null}(\partial_p^*)$$

Proof: $$\sum_{i \in I} \lambda_i(\mathcal{T}_p) = \dim \mathrm{im}\, \partial_p^* = \dim C_{p-1}(K) - \dim \ker \partial_p^*$$

Corollary

For the orthonormal basis of $C_p(K)$, $\{\sqrt{w(\sigma)}\,\sigma : \sigma \in K_p\}$, we have

$$\sum_{i \in I} \lambda_i(\mathcal{T}_p) = \sum_{\sigma} \mathcal{R}_p(\sqrt{w(\sigma)}\,\sigma, \sqrt{w(\sigma)}\,\sigma) = \sum_{\sigma} w(\sigma)\, r_\sigma = |K_{p-1}| - \mathrm{null}(\partial_p^*)$$

which for $p = 1$ recovers Foster's theorem.

Simplicial spanning forests and trees

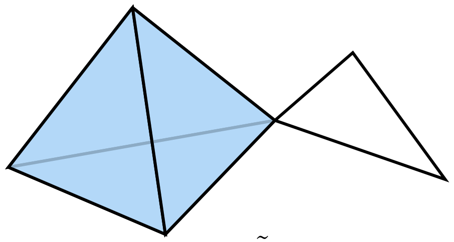
Simplicial spanning forests

Let K be a d-dimensional complex and $T \subset K$ a p-dimensional subcomplex, $0 < p \leq d$. T is a *simplicial spanning forest* if:

1. $T_{\leq p-1} = K_{\leq p-1}$,
2. $\tilde{\beta}_p(T) = 0$.

Proposition

A spanning forest has *maximal* number of p-simplices if and only if $\tilde{\beta}_{p-1}(T) = \tilde{\beta}_{p-1}(K)$.

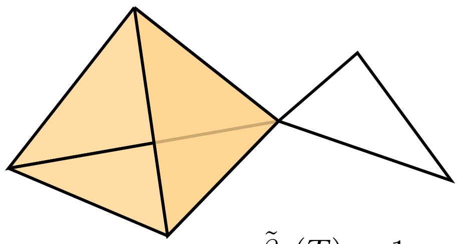


Simplicial complex

$\tilde{\beta}_0(K) = 1$

$\tilde{\beta}_1(K) = 1$

$\tilde{\beta}_2(K) = 1$



Maximal spanning forest

$\tilde{\beta}_0(T) = 1$

$\tilde{\beta}_1(T) = 1$

$\tilde{\beta}_2(T) = 0$

Simplicial spanning forests and trees

Simplicial spanning trees

Let K be a d-dimensional complex and $T \subset K$ a p-dimensional subcomplex, $0 < p \leq d$.
T is a *simplicial spanning tree* if:

1. $T_{\leq p-1} = K_{\leq p-1}$,
2. $\tilde{\beta}_p(T) = 0$, *(acyclicity)*
3. $\tilde{\beta}_{p-1}(T) = 0$. *(connectedness)*

Let $\mathbb{T}_p(K)$ be the set of p-dimensional simplicial spanning trees in K.

- You can substitute 2. or 3. above by

$$|T_p| = |K_p| - \tilde{\beta}_p(K_{\leq p}) - \tilde{\beta}_{p-1}(K_{\leq p})$$

which represents a slight departure from the original definition by Duval et al., (2009).

- If $\tilde{\beta}_{p-1}(K) = 0$, then being a spanning tree is equivalent to being a maximal spanning forest.

Kirchhoff's matrix theorem for simplicial complexes

Theorem

Let K be a weighted d-dimensional simplicial complex, $d \geq 2$, with $\tilde{\beta}_{d-1}(K) = \tilde{\beta}_{d-2}(K) = 0$. Set

$$\tau_d(K, w) := \sum_{T \in \mathbb{T}_d(K)} |\tilde{H}_{d-1}(T; \mathbb{Z})|^2 \prod_{\sigma \in T_d} w(\sigma).$$

For $\sigma \in K_d$, set

$$\tau_d(K, w)_\sigma := \sum_{\substack{T \in \mathbb{T}_d(K) \\ \sigma \in T}} |\tilde{H}_{d-1}(T; \mathbb{Z})|^2 \prod_{\rho \in T_d} w(\rho).$$

Then

$$w(\sigma)\, r_\sigma = \frac{\tau_d(K, w)_\sigma}{\tau_d(K, w)}.$$

If $d = 1$ and $H_0(T; \mathbb{Z}) = 0$ (connected trees), then:

- $\tau_1(K, w)$ is the weighted count of trees in K,
- $\tau_1(K, w)_\sigma$ is the weigthed count of trees in K that contain σ.

Kirchhoff's matrix theorem for simplicial complexes

Corollary

If we sample a tree with probability

$$\Pr[\mathbf{T} = T] = \frac{|\tilde{H}_{d-1}(T;\mathbb{Z})|^2 \prod_{\rho \in T} w(\rho)}{\sum_{T' \in \mathbb{T}_d(K)} |\tilde{H}_{d-1}(T';\mathbb{Z})|^2 \prod_{\rho \in T'} w(\rho)}$$

then, the probability of finding the d-simplex σ in such tree is precisely

$$\Pr[\sigma \in \mathbf{T}] = w(\sigma)\, r_\sigma .$$

For $d = 1$, $|\tilde{H}_0(T;\mathbb{Z})| = 1$ for any tree, and this recovers the result for graphs.

Conclusion and future work

What we have so far:

- Basis-free definition of effective resistance unifying existing approaches
- Extension of effective resistance to higher dimensions and chains
- Metric properties, Foster's theorem, and Kirchhoff's theorem, all valid in greater generality

What is next:

- Connect effective resistance with simplicial random walks
- Simplicial resistance curvature (cf. Devriendt and Lambiotte, 2021)
- Implement simplicial effective resistance and study its behaviour in real datasets (e.g. MANTRA)

Thanks to the Bernoulli Center at EPFL for hosting WinCompTop3 and ICERM for hosting the *Finding Geometric and Topological Cores of Higher Graphs* collaborative program

Thanks for your attention!