The Pareto Grid in Multiparameter Persistence

Based in joint work with F. Conti, P. Frosini, U. Fugacci, N. Quercioli,

S. Scaramuccia and F. Tombari

Eloy Mosig García

March 24, 2026

Table of Contents

Introduction

- Let M be a closed smooth Riemannian manifold of dimension ≥ 2.
- Consider a smooth function $\varphi = (\varphi_1, \varphi_2) \colon M \to \mathbb{R}^2$.
- Taking the homology of the sublevel sets of φ yields a biparameter persistence module.

- Let M be a closed smooth Riemannian manifold of dimension ≥ 2.
- Consider a smooth function $\varphi = (\varphi_1, \varphi_2)\colon M \to \mathbb{R}^2$.
- Taking the homology of the sublevel sets of φ yields a biparameter persistence module.
- **Obstruction:** There is no complete invariant for n-parameter persistence for $n \geq 2$.

- Let M be a closed smooth Riemannian manifold of dimension ≥ 2.
- Consider a smooth function $\varphi = (\varphi_1, \varphi_2)\colon M \to \mathbb{R}^2$.
- Taking the homology of the sublevel sets of φ yields a biparameter persistence module.
- **Obstruction:** There is no complete invariant for n-parameter persistence for $n \geq 2$.
- **Strategy:** ”Slice” the biparameter module with a suitable family of monoparameter modules.

1-parameter ”slice” of a filtration

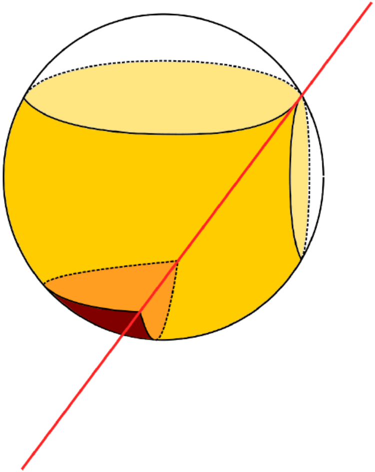


Table of Contents

2 The Pareto Grid

Pareto Critical points

Definition: Jacobi set

The Jacobi set, $\mathbb{J}(\varphi)$, is the set of points $x \in M$ at which the gradients $\nabla\varphi_1(x)$ and $\nabla\varphi_2(x)$ are parallel.

$$\exists(\lambda_1, \lambda_2) \neq (0,0) \text{ s.t. } \lambda_1\nabla\varphi_1(x) + \lambda_2\nabla\varphi_2(x) = 0$$

Definition: Jacobi set

The Jacobi set, $\mathbb{J}(\varphi)$, is the set of points $x \in M$ at which the gradients $\nabla\varphi_1(x)$ and $\nabla\varphi_2(x)$ are parallel.

$$\exists(\lambda_1, \lambda_2) \neq (0,0) \text{ s.t. } \lambda_1\nabla\varphi_1(x) + \lambda_2\nabla\varphi_2(x) = 0$$

Pareto-critical set

The Pareto critical set, $\mathbb{J}_P(\varphi)$, is the subset of $\mathbb{J}(\varphi)$ consisting of points $x \in M$ at which the gradients $\nabla\varphi_1(x)$ and $\nabla\varphi_2(x)$ are parallel and of opposing direction.

$$\exists\lambda_1, \lambda_2 \geq 0 \text{ s.t. } \lambda_1\nabla\varphi_1(x) + \lambda_2\nabla\varphi_2(x) = 0$$

Pareto Critical points

2 The Pareto Grid

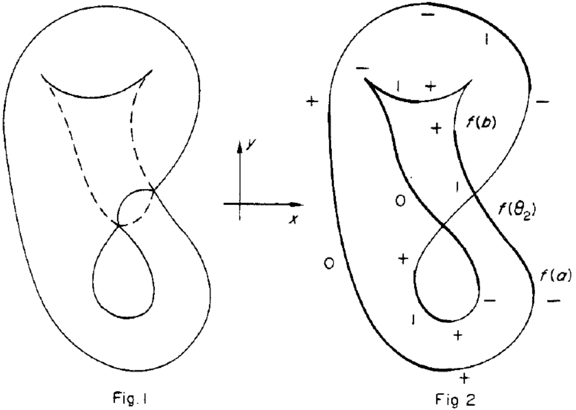


Figure: Image courtesy of [Wan'75]

Generic assumptions

Assume φ satisfies the following:

- No point $x \in M$ exists where both $\nabla\varphi_1(x)$ and $\nabla\varphi_2(x)$ vanish simultaneously.

Generic assumptions

Assume φ satisfies the following:

- No point $x \in M$ exists where both $\nabla\varphi_1(x)$ and $\nabla\varphi_2(x)$ vanish simultaneously.
- $\mathbb{J}(\varphi)$ is a 1-dimensional manifold smoothly embedded in M, consisting of finitely many circle components.
- $\mathbb{J}_P(\varphi)$ is a 1-dimensional closed submanifold of $\mathbb{J}(\varphi)$ with boundary.

Generic assumptions

Assume φ satisfies the following:

- No point $x \in M$ exists where both $\nabla\varphi_1(x)$ and $\nabla\varphi_2(x)$ vanish simultaneously.
- $\mathbb{J}(\varphi)$ is a 1-dimensional manifold smoothly embedded in M, consisting of finitely many circle components.
- $\mathbb{J}_P(\varphi)$ is a 1-dimensional closed submanifold of $\mathbb{J}(\varphi)$ with boundary.
- The set of cusp points $\mathbb{J}_C(\varphi)$ splits $\mathbb{J}_P(\varphi)$ into finitely many connected components diffeomorphic to intervals. With respect to any parametrization, one component of φ is decreasing and the other one is increasing along these connected components.

Generic assumptions

Assume φ satisfies the following:

- No point $x \in M$ exists where both $\nabla\varphi_1(x)$ and $\nabla\varphi_2(x)$ vanish simultaneously.
- $\mathbb{J}(\varphi)$ is a 1-dimensional manifold smoothly embedded in M, consisting of finitely many circle components.
- $\mathbb{J}_P(\varphi)$ is a 1-dimensional closed submanifold of $\mathbb{J}(\varphi)$ with boundary.
- The set of cusp points $\mathbb{J}_C(\varphi)$ splits $\mathbb{J}_P(\varphi)$ into finitely many connected components diffeomorphic to intervals. With respect to any parametrization, one component of φ is decreasing and the other one is increasing along these connected components.

These assumptions are generic on the set of smooth functions $M \to \mathbb{R}^2$ [Wan'75, EH'04].

Definition: Pareto Grid

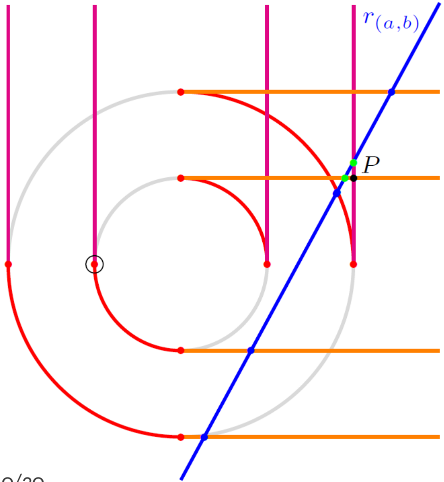
The Pareto grid $\Gamma(\varphi)$ of φ is the image of the Pareto-critical points $\varphi(\mathbb{J}_P(\varphi)) \subset \mathbb{R}^2$.

Definition: Extended Pareto Grid

Let $\{(x_i^1, y_i^1)\}_i$ and $\{(x_j^2, y_j^2)\}_j$ denote the (finite) critical points of φ_1 and φ_2, respectively. The Extended Pareto grid $\bar{\Gamma}(\varphi)$ consists of the union

$$\bar{\Gamma}(\varphi) = \Gamma(\varphi) \cup \bigcup_i v_i \cup \bigcup_j h_j,$$

where $v_i = \{(x_i^1, y) \mid y \geq y_i^1\}$ and $h_j = \{(x, y_j^2) \mid x \geq x_j^2\}$.



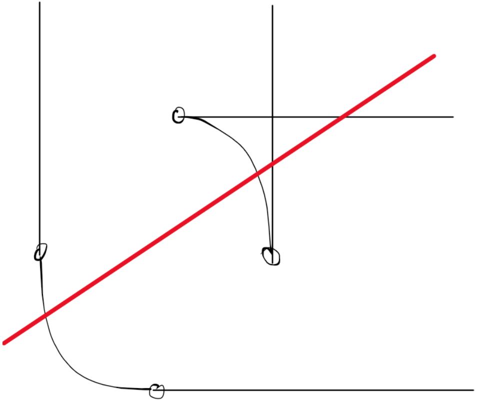
Contours

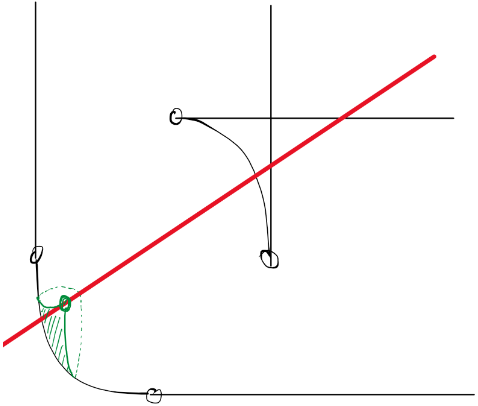
The closures of the connected components of $\mathbb{J}_P(\varphi) \setminus \mathbb{J}_C(\varphi)$ under φ are called *proper contours*. The half-lines in $\bar{\Gamma}(\varphi)$ are the *improper contours*.

Interesting properties of the Pareto Grid

- **Finiteness:** The number of contours is finite.
- **Unique Intersections:** Because filtering lines $r_{(a,b)}$ always have a strictly positive slope, their intersection with any contour is either empty or exactly a single point.

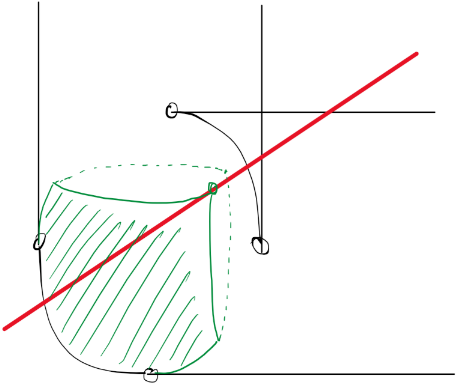
A badly drawn example

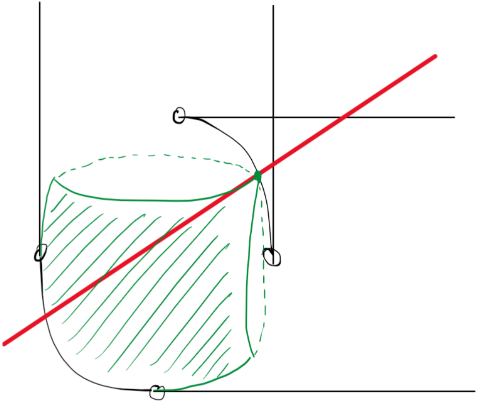




A badly drawn example

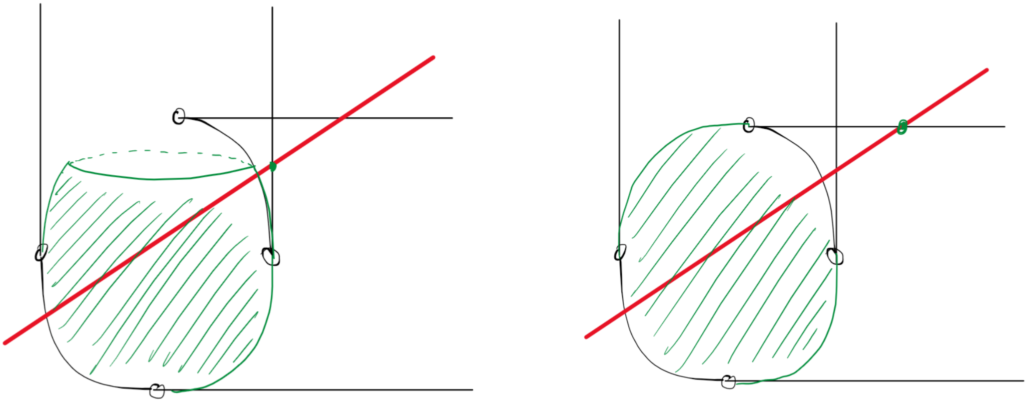
2 The Pareto Grid





A badly drawn example

2 The Pareto Grid



A properly drawn example

2 The Pareto Grid

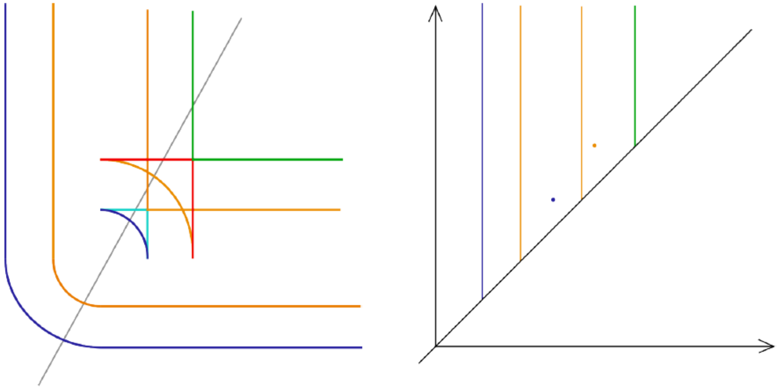


Figure: The Pareto grid of the projection of a torus together with the corresponding persistent diagram(s).

Table of Contents

3 Distances and GENEOs

1-Parameter Reductions: GENEOs [Fro'16]

Given a set X let $\Phi \subset \{X \to \mathbb{R}^k\}$ and $G < Homeo(X)$ acting on Φ by right composition. The pair (Φ, G) is called a perception pair.
Consider two perception pairs (Φ, G) and (Ψ, H) and fix a homomorphism $T\colon G \to H$. A *Group Equivariant Non Expansive Operator (GENEO)* is a function $F\colon \Phi \to \Psi$ such that:

- $F(\varphi \circ g) = F(\varphi) \circ T(g)$,
- $D_\Psi(F(\varphi), F(\varphi')) \leq D_\Phi(\varphi, \varphi')$.

Given a set X let $\Phi \subset \{X \to \mathbb{R}^k\}$ and $G < Homeo(X)$ acting on Φ by right composition. The pair (Φ, G) is called a perception pair.
Consider two perception pairs (Φ, G) and (Ψ, H) and fix a homomorphism $T\colon G \to H$. A *Group Equivariant Non Expansive Operator (GENEO)* is a function $F\colon \Phi \to \Psi$ such that:

- $F(\varphi \circ g) = F(\varphi) \circ T(g)$,
- $D_\Psi(F(\varphi), F(\varphi')) \leq D_\Phi(\varphi, \varphi')$.

Given a parametrized family of GENEOs $\mathcal{F} = \{F_\theta\}_\theta$ one can define:

$$D_{\mathcal{F}}(\varphi, \psi) = \sup_\theta d_B(Dgm_k(F_\theta(\varphi)), Dgm_k(F_\theta(\psi))).$$

Stability [BFGQ'19]

$D_{\mathcal{F}}$ is stable w.r.t. the supremum norm $\|\cdot\|_{\infty}$.

Stability [BFGQ'19]

$D_{\mathcal{F}}$ is stable w.r.t. the supremum norm $\|\cdot\|_\infty$.

Two examples with $\Phi \subset (\mathbb{R}^2)^X$, $G, H = 0$ and T the identity homomorphism:

Stability [BFGQ'19]

$D_{\mathcal{F}}$ is stable w.r.t. the supremum norm $\|\cdot\|_\infty$.

Two examples with $\Phi \subset (\mathbb{R}^2)^X$, $G, H = 0$ and T the identity homomorphism:

1. Foliation Method (Classical Matching Distance MD):
For $(a, b) \in]0, 1[\times \mathbb{R}$:

$$\varphi^*_{(a,b)}(x) = \min\{a, 1-a\} \max\left\{\frac{\varphi_1(x) - b}{a}, \frac{\varphi_2(x) + b}{1-a}\right\}.$$

Stability [BFGQ'19]

$D_{\mathcal{F}}$ is stable w.r.t. the supremum norm $\|\cdot\|_\infty$.

Two examples with $\Phi \subset (\mathbb{R}^2)^X$, $G, H = 0$ and T the identity homomorphism:

1. **Foliation Method** (Classical Matching Distance MD):

For $(a, b) \in]0, 1[\times\mathbb{R}$:

$$\varphi^*_{(a,b)}(x) = \min\{a, 1-a\} \max\left\{\frac{\varphi_1(x) - b}{a}, \frac{\varphi_2(x) + b}{1-a}\right\}.$$

2. **Convex combination** (Convex Matching Distance CMD):

For $t \in [0, 1]$:

$$\varphi^t(x) = (1-t)\varphi_1(x) + t\varphi_2(x).$$

Advantages of the CMD

- **Differentiability Preservation:** For each t, $\varphi \mapsto \varphi^t$ preserves the smoothness of the original functions. In contrast, the $\min / \max$ operators in $\varphi^*_{(a,b)}$ only preserve continuity.
- **Dimensionality Reduction:** CMD is defined via a 1-parameter family.

Table of Contents

4 Position Theorems

Position Theorem (MD) [CEF'19]

Let $(a, b) \in]0, 1[\times \mathbb{R}$ and let w be a finite coordinate of a point in $Dgm_k(\varphi^*_{(a,b)}) \setminus \{\Delta\}$. Then there exists a point $P = (x, y) \in r_{(a,b)} \cap \bar{\Gamma}(\varphi)$ such that:

- $w = \min\left\{1, \frac{1-a}{a}\right\}(x - b)$ if $a \in [0, 1[$
- $w = \min\left\{1, \frac{a}{1-a}\right\}(y + b)$ if $a \in]0, 1]$

Geometric Intuition: The finite coordinates of the diagram are found by intersecting the filtering line $r_{(a,b)}$ with the Extended Pareto Grid $\bar{\Gamma}(\varphi)$, and then applying a scaling factor that depends on the slope of the line.

Position Theorem (CMD) [CFFMQST'26]

Let $t \in [0, 1]$ and let w be a finite coordinate of a point in $Dgm_k(\varphi^t) \setminus \{\Delta\}$. Then there exist a contour $\alpha : [0, 1] \to \mathbb{R}^2$ and a $\bar{\tau} \in [0, 1]$ such that:

1. $\alpha(\bar{\tau}) \cdot (1 - t, t) = w$
2. $\frac{d\alpha}{d\tau}(\bar{\tau}) \cdot (1 - t, t) = 0$

Geometric Intuition: The finite coordinates of the diagram are found by taking lines with direction $(1 - t, t)$ and finding where they intersect the contours orthogonally.

Proof Sketch of the Position Theorem (CMD)

- **Step 1:** Since w is a finite coordinate in the persistence diagram $Dgm_k(\varphi^t)$, w is a critical value for φ^t.

Proof Sketch of the Position Theorem (CMD)

- **Step 1:** Since w is a finite coordinate in the persistence diagram $Dgm_k(\varphi^t)$, w is a critical value for φ^t.
- **Step 2:** Thus, there exists $\bar{x} \in M$ such that $w = \varphi^t(\bar{x})$ and $\nabla \varphi^t(\bar{x}) = 0$.

- **Step 1:** Since w is a finite coordinate in the persistence diagram $Dgm_k(\varphi^t)$, w is a critical value for φ^t.
- **Step 2:** Thus, there exists $\bar{x} \in M$ such that $w = \varphi^t(\bar{x})$ and $\nabla\varphi^t(\bar{x}) = 0$.
- **Step 3:** Expanding the gradient gives $(1-t)\nabla\varphi_1(\bar{x}) + t\nabla\varphi_2(\bar{x}) = 0$, proving $\bar{x}$ is a Pareto critical point of φ. Let $\alpha(\bar{\tau}) = \varphi(\bar{x})$. Thus $w = \alpha(\bar{\tau}) \cdot (1-t, t)$.

Proof Sketch of the Position Theorem (CMD)

- **Step 1:** Since w is a finite coordinate in the persistence diagram $Dgm_k(\varphi^t)$, w is a critical value for φ^t.
- **Step 2:** Thus, there exists $\bar{x} \in M$ such that $w = \varphi^t(\bar{x})$ and $\nabla\varphi^t(\bar{x}) = 0$.
- **Step 3:** Expanding the gradient gives $(1-t)\nabla\varphi_1(\bar{x}) + t\nabla\varphi_2(\bar{x}) = 0$, proving $\bar{x}$ is a Pareto critical point of φ. Let $\alpha(\bar{\tau}) = \varphi(\bar{x})$. Thus $w = \alpha(\bar{\tau}) \cdot (1-t, t)$.
- **Step 4:** For a regular parametrisation γ of α, we have $0 = \nabla\varphi^t(\bar{x}) \cdot \frac{d\gamma}{d\tau}(\bar{\tau})$. Applying the chain rule, this becomes $\frac{d\alpha}{d\tau}(\bar{\tau}) \cdot (1-t, t) = 0$.

Special Values

- The Position Theorem(s) translate the computation of MD (CMD) into tracking the intersections between filtering lines and $\bar{\Gamma}(\varphi)$ ($\Gamma(\varphi)$, respectively).
- **Obstruction:** The matching realizing the Bottleneck distance is not continuous with respect to θ.

Special values

The sudden shifts in the optimal matching occur at a privileged subset of parameters θ. To compute the distance, we can restrict our search to the **special set** $Sp(\varphi, \psi)$.

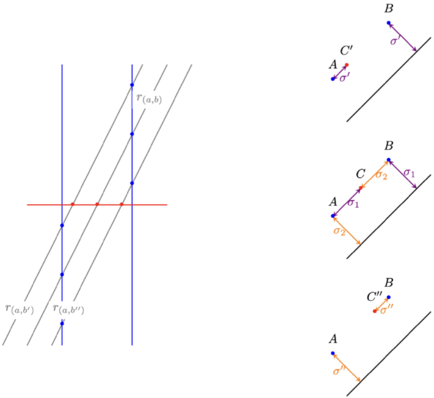


Figure: A configuration on $\bar{\Gamma}(\varphi)$ corresponding to a special value.

Main Theorems: Realizing the Distances

Theorem for MD [EFQT'23],[FMQT'25]

If (a, b) realises MD(φ, ψ), then either $a = \frac{1}{2}$ or $(a, b) \in Sp^{MD}(\varphi, \psi)$.

Theorem for CMD [CFFMQST'26]

If t realises CMD(φ, ψ), then either $t \in Sp^{CMD}(\varphi, \psi)$.

Table of Contents

5 Conclusion

Conclusions and Future Directions

Summary

- The **Pareto grid** is a partial invariant of filtrations arising from $\mathbb{R}^2$-valued functions closely related to its asociated monoparametric filtrations.
- This relation is explicited via the Position theorem, allowing computation.

Open Problems

- Identify the minimal hypothesis under which the special set $Sp(\varphi, \psi)$ is generically.
- Develop and optimize algorithmic implementations that exploit the Special set set to compute the MD/CMD efficiently.

Bibliography

5 Conclusion

Y. H. Wan.
Morse theory for two functions.
Topology, 14(3):217–228, 1975.

A. Cerri, M. Ethier, and P. Frosini.
On the geometrical properties of the coherent matching distance in 2D persistent homology.
J. Appl. Comput. Topol., 3:381–422, 2019.

M. G. Bergomi, P. Frosini, D. Giorgi, and N. Quercioli.
Towards a topological-geometrical theory of group equivariant non-expansive operators.
Nat. Mach. Intell., 1:423–433, 2019.

P. Frosini, E. Mósig García, N. Quercioli, and F. Tombari.
Matching distance via the extended Pareto grid.
SIAM J. Appl. Algebra Geom., 9(3):554–576, 2025.

F. Conti, P. Frosini, U. Fugacci, E. Mósig García, N. Quercioli, S. Scaramuccia, and F. Tombari.
The Convex Matching Distance in Multiparameter Persistence.
Preprint, arXiv:2512.02944v2, 2026.

Thank you for your attention!

Check out my site!

Backup: The Special Set for MD

Definition [FMQT'25]

The special set $Sp(\varphi, \psi)$ for the MD is the collection of all (a, b) in $]0, 1[\times\mathbb{R}$ for which there exist two different pairs of contours $\{\alpha_P, \alpha_Q\}$, $\{\alpha_R, \alpha_S\}$ in $Ctr(\varphi, \psi)$ intersecting the line $r_{(a,b)}$ at points P, Q, R, S respectively, such that:

Case 1 ($a \leq 1/2$):

$$c_1|x_P - x_Q| = c_2|x_R - x_S| \quad \text{with } c_1, c_2 \in \{1, 2\}$$

Case 2 ($a \geq 1/2$):

$$c_1|y_P - y_Q| = c_2|y_R - y_S| \quad \text{with } c_1, c_2 \in \{1, 2\}$$

Intuition: The bottleneck distances associated with distinct pairs of homological features become proportional, causing a tie in the optimal matching.

Backup: The Special Set for CMD

Definition [CFFMQST'25]

The special set $Sp(\varphi, \psi)$ for CMD consists of values $t \in [0, 1]$ satisfying at least one of the following:

1. A line with direction $(1 - t, t)$ intersects a contour orthogonally at one of its endpoints.
2. Lines r_i intersect contours α_i orthogonally at P_i ($i \in \{1, 2, 3, 4\}$) such that:

$$(P_1 - P_2) \cdot (1 - t, t) = c\big((P_3 - P_4) \cdot (1 - t, t)\big), \quad c \in \{1, 2\}$$

3. Orthogonal intersections P_1, P_2 exist where either a contour has zero curvature, their signed radii l_1, l_2 are equal, or t satisfies:

$$t = \frac{\zeta}{1 + \zeta} \quad \text{with} \quad \zeta = \tan\left(\frac{1}{2}\arcsin\left(1 - \left(\frac{(y_1 - y_2) - (x_1 - x_2)}{l_1 - l_2}\right)^2\right)\right)$$