

# Spaces of persistence diagrams and beyond

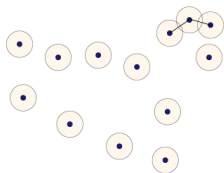
Mauricio Che

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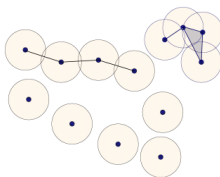
May 2025

# Persistence diagrams

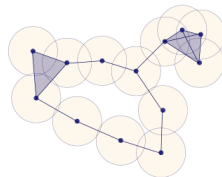
# Topological Data Analysis pipeline



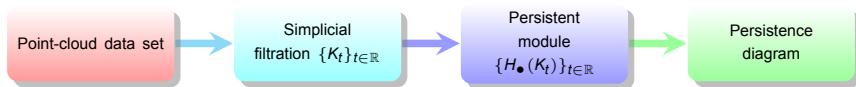
(a) Rips complex at some scale



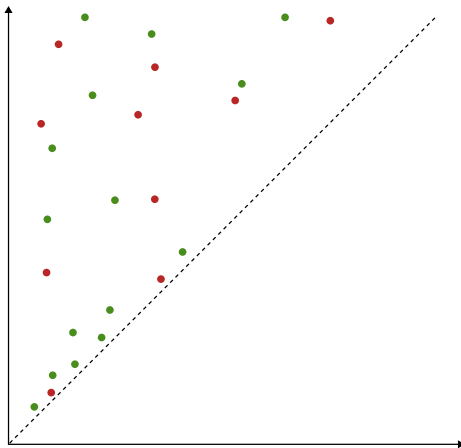
(b) ..at a larger scale.

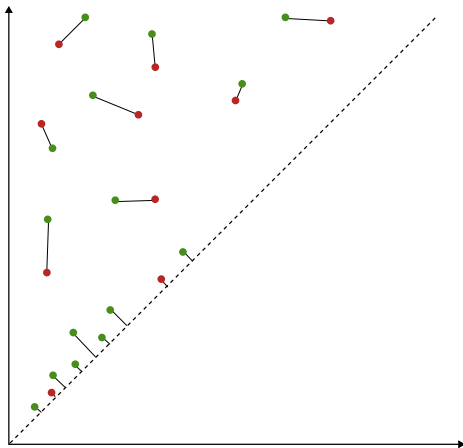


(c) ...and at an even larger scale.



- **Persistence diagrams** are (countable multi-)sets of points  $\{(a_i, b_i)\}_{i \in I}$  in  $\mathbb{R}^2$ , where each  $(a_i, b_i)$  represents a persistent homology class whose 'birth' and 'death' occur at times  $a_i$  and  $b_i$ , respectively.





- **Persistence diagrams** are (countable multi-)sets of points  $\{(a_i, b_i)\}_{i \in I}$  in  $\mathbb{R}^2$ , where each  $(a_i, b_i)$  represents a persistent homology class whose ‘birth’ and ‘death’ occur at times  $a_i$  and  $b_i$ , respectively.
- The  $L^p$ -**Wasserstein distance** (or **Bottleneck distance**, when  $p = \infty$ ) between two persistence diagrams  $\sigma$  and  $\tau$  is given by

$$d_p(\sigma, \tau) = \inf_{\phi} \left( \sum_{x \in \tilde{\sigma}} \|x - \phi(x)\|^p \right)^{1/p}$$

where the infimum runs over bijections  $\phi$  between diagrams  $\tilde{\sigma}$  and  $\tilde{\tau}$ , which coincide with  $\sigma$  and  $\tau$  up to adding enough points in  $\Delta = \{(x, y) \in \mathbb{R}^2 : x = y\}$ .

### Theorem (Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot, '09)

Let  $X$  and  $Y$  be a finite metric spaces. Then,

$$d_{\infty}(\text{dgm}_k(\text{VR}(X)), \text{dgm}_k(\text{VR}(Y))) \leq 2d_{\text{GH}}(X, Y).$$

### Theorem (Cohen-Steiner, Edelsbrunner, Harer, '07)

Let  $X$  be a triangulable compact metric space and  $f, g: X \rightarrow \mathbb{R}$  be sufficiently nice functions. Then,

$$d_{\infty}(\text{dgm}_k(f), \text{dgm}_k(g)) \leq \|f - g\|_{\infty}.$$

### Theorem (Cohen-Steiner, Edelsbrunner, Harer, Mileyko, '10)

Let  $X$  be a triangulable, compact metric sufficiently nice space, and let  $f, g: X \rightarrow \mathbb{R}$  be sufficiently nice Lipschitz functions. Then

$$d_p(\text{dgm}_k(f), \text{dgm}_k(g)) \leq C\|f - g\|_{\infty}^{1-p/k}$$

for all  $p \geq k$  and some constant  $C = C(X, k, p, \text{Lip}(f), \text{Lip}(g))$ .

- For each  $p \in [1, \infty]$  we obtain the corresponding  $L^p$ -space of persistence diagrams

$$\mathcal{D}_p = \left\{ \sigma : \sum_{x \in \sigma} \text{dist}(x, \Delta)^p < \infty \right\}.$$

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- Optimal partial bijections between persistence diagrams exist (Mileyko, Mukherjee, Harer, '11).
- These spaces are complete, separable, and geodesic (Mileyko, Mukherjee, Harer, '11). Moreover, when  $p \in (1, \infty)$ , geodesics in  $\mathcal{D}_p$  are given by interpolations along geodesics in  $\mathbb{R}^2$ . Moreover,  $\mathcal{D}_p$  is non-branching (Chowdhury, '19).

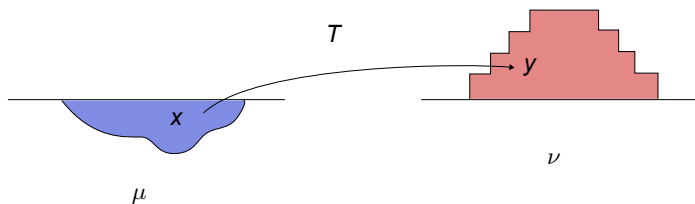
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- When  $p = 2$ ,  $\mathcal{D}_2$  is a non-negatively curved Alexandrov space (defined later) (Turner, Mileyko, Mukherjee, Harer, '14).

# Beyond, vol. I: Optimal transport

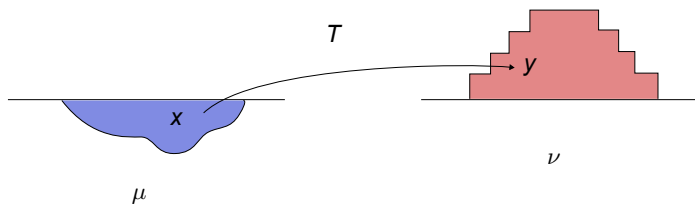
# Optimal transport: Monge



Problem: Given probability measures  $\mu, \nu$ , find  $T: X \rightarrow X$  such that

$$\begin{cases} T_{\#}\mu = \nu \\ \int c(x, T(x)) d\mu(x) \text{ is minimal} \end{cases}$$

# Optimal transport: Kantorovich



Problem: Given probability measures  $\mu, \nu$ , find a measure  $\pi$  in  $X^2$  such that

$$\begin{cases} p_{\#}^1 \pi = \mu, p_{\#}^2 \pi = \nu \\ \int c(x, y) d\pi(x, y) \text{ is minimal} \end{cases}$$

- Let  $X$  be a topological space and consider the set

$$\mathcal{P}(X) = \{\text{Borel probability measures on } X\}.$$

For any  $\mu, \nu \in \mathcal{P}(X)$  denote by

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{P}(X^2) : p_{\#}^1 \pi = \mu, p_{\#}^2 \pi = \nu\}$$

the set of **transport plans** between  $\mu$  and  $\nu$ .

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the set of **transport plans** between  $\mu$  and  $\nu$ .

- For any measurable **cost function**  $c: X^2 \rightarrow [0, \infty]$ , the associated **optimal transport problem** consists of minimising the **total cost**,

$$\pi \mapsto \int_{X^2} c(x, y) d\pi(x, y),$$

over  $\pi \in \Pi(\mu, \nu)$ .

## Theorem

Assuming that  $X$  is a completely metrizable and separable topological space,  $\mu, \nu \in \mathcal{P}(X)$ , and  $c: X^2 \rightarrow [0, \infty]$  is lower semi-continuous, the set of **optimal plans**,

$$\Pi^{\text{opt}}(\mu, \nu) = \operatorname{argmin} \left\{ \pi \mapsto \int_{X^2} c(x, y) d\pi(x, y) : \pi \in \Pi(\mu, \nu) \right\}$$

is non-empty.

- Let  $(X, d)$  be a complete and separable metric space. Let  $c(x, y) = d(x, y)^p$  for some  $p \in [1, \infty)$ . Then the  $L^p$ -**Wasserstein distance** between  $\mu, \nu \in \mathcal{P}(X)$  is given by:

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{X^2} d(x, y)^p d\pi(x, y) \right)^{1/p}.$$

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- The function  $W_p$  defines a metric on the set

$$\mathcal{P}_p(X) = \{ \mu \in \mathcal{P}(X) : W_p(\mu, \delta_x) < \infty \text{ for some (thus for any) } x \in X \}.$$

This is the  **$L^p$ -Wasserstein space** over  $X$ .

## Theorem

*Let  $X$  be a complete and separable metric space and  $p \in [1, \infty)$ . Then optimal plans exist in  $\mathcal{P}_p(X)$ . Moreover  $\mathcal{P}_p(X)$  is complete and separable.*

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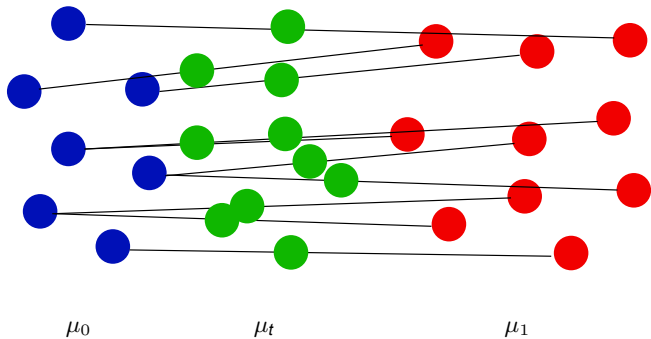
## Theorem (McCann, '97; Lisini, '07; Villani, '09; Ambrosio, Gigli, '13)

Let  $X$  be complete, separable, and geodesic, then  $\mathcal{P}_p(X)$  is geodesic. Moreover, if  $p \in (1, \infty)$  and  $(\mu_t)_{t \in [0,1]}$  is a continuous curve in  $\mathcal{P}_p(X)$ , then the following conditions are equivalent

- $(\mu_t)_{t \in [0,1]} \in \text{Geod}(\mathcal{P}_p(X))$ .
- There exists  $\mu \in \mathcal{P}(\text{Geod}(X))$  such that:

$$\begin{cases} (\mathbf{e}_0, \mathbf{e}_1)_{\#}(\mu) \in \Pi^{\text{opt}}(\mu_0, \mu_1) \\ (\mathbf{e}_t)_{\#}(\mu) = \mu_t \text{ for all } t \in [0, 1]. \end{cases}$$

Furthermore, if  $X$  is non-branching then  $\mathcal{P}_p(X)$  is non-branching.

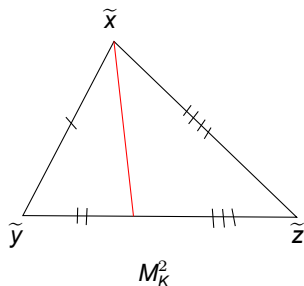
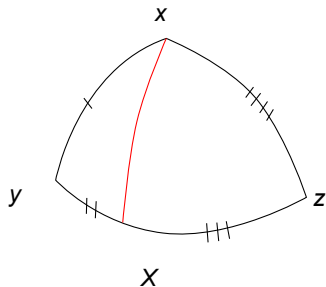


Theorem (Otto, '01; Ambrosio, Gigli, Savaré, '05; Sturm, '06)

*If  $X$  is an **Alexandrov space** with non-negative curvature then  $\mathcal{P}_2(X)$  is also an Alexandrov space with non-negative curvature.*

# Alexandrov spaces

Given  $K \in \mathbb{R}$ , an **Alexandrov space of curvature bounded below by  $K$**  is a geodesic space  $X$  such that any point  $p \in X$  has a neighbourhood  $U_p$  such that triangles in  $U_p$  are thicker than triangles in the model surface of constant curvature  $K$ :



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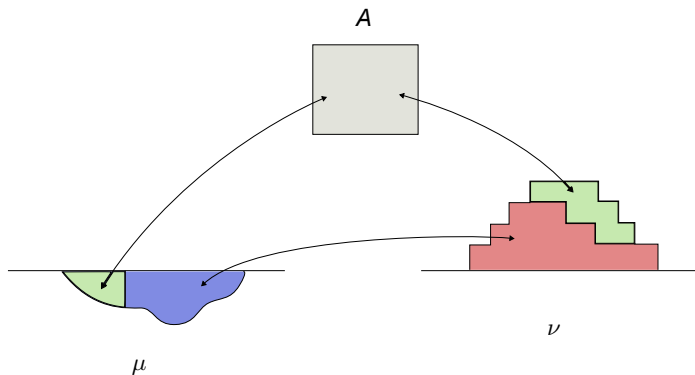
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- Many smooth objects have generalisations in Alexandrov geometry, e.g. tangent cones, spaces of directions, coordinates, etc.

# Beyond, vol. II: Optimal partial transport & generalised spaces of persistence diagrams

# Optimal partial transport



- Let  $X$  be a *proper* metric space and  $A$  be a non-empty closed subset of  $X$ . We consider the set

$$\mathcal{M}(X, A) = \{\text{Radon measures on } X \setminus A\}.$$

Recall that a Radon measure, in this context, is a non-negative Borel measure that is finite on bounded sets.

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- Given  $\mu, \nu \in \mathcal{M}(X, A)$  denote by

$$\tilde{\Pi}(\mu, \nu) = \{\pi \in \mathcal{M}(X^2, A^2) : p_{\#}^1 \pi|_{X \setminus A} = \mu, p_{\#}^2 \pi|_{X \setminus A} = \nu\}$$

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the set of **partial transport plans** between  $\mu$  and  $\nu$ .

- For any  $\mu, \nu \in \mathcal{M}(X, A)$ , we define the  $L^p$ -**partial Wasserstein distance** by

$$\tilde{W}_p(\mu, \nu) = \inf_{\pi \in \tilde{\Pi}(\mu, \nu)} \left( \int_{X^2 \setminus A^2} d(x, y)^p d\pi(x, y) \right)^{1/p}$$

## Theorem (Figalli, Gigli, '10; Divol, Lacombe, '19; C., '24)

Let  $X$  be a proper metric space,  $A \subset X$  closed and non-empty. Let

$$\mathcal{M}_p(X, A) = \left\{ \mu \in \mathcal{M}(X, A) : \int_{X \setminus A} d(x, A)^p \mu(x) < \infty \right\}.$$

Then the following conditions hold:

- Optimal partial transport plans exist.
- $\mathcal{M}_p(X, A)$  is a complete and separable metric space.

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## Theorem (Figalli, Gigli, '10; C., '24)

If  $X$  is proper and geodesic, then  $\mathcal{M}_p(X, A)$  is geodesic too. Moreover, if  $p \in (1, \infty)$  then for any continuous curve  $(\mu_t)_{t \in [0, 1]}$  in  $\mathcal{M}_p(X, A)$  the following are equivalent:

- $(\mu_t)_{t \in [0, 1]}$  is a geodesic.
- There exists  $\mu \in \mathcal{M}(\text{Geod}(X))$  such that  $(e_0, e_1)_{\#} \mu \in \tilde{\Pi}^{\text{opt}}(\mu, \nu)$  and  $(e_t)_{\#} \mu|_{X \setminus A} = \mu_t$  for all  $t \in [0, 1]$ .

Furthermore, if  $X$  is non-branching, then  $\mathcal{M}_p(X, A)$  is non-branching.

## Theorem (C., '24)

*Let  $X$  be a proper, non-negatively curved Alexandrov space, and  $A \subset X$  closed and non-empty. Then  $\mathcal{M}_2(X, A)$  is a non-negatively curved Alexandrov space as well.*

## Definition

Given a metric space  $X$  and a closed, non-empty subset  $A \subset X$ , let

$$\mathcal{D}(X, A) := \{\tilde{\sigma} : \tilde{\sigma} \text{ is a countable collections of points in } X\} / \sim$$

where  $\tilde{\sigma} \sim \tilde{\tau}$  if  $\tilde{\sigma}$  and  $\tilde{\tau}$  coincide outside of  $A$ . If  $p \in [1, \infty)$  and  $\sigma, \tau \in \mathcal{D}(X, A)$ , let

$$d_p(\sigma, \tau) = \inf_{\phi} \left( \sum_{x \in \tilde{\sigma}} d(x, \phi(x))^p \right)^{1/p}$$

where  $\phi$  runs over all partial bijections between representatives  $\tilde{\sigma}$  and  $\tilde{\tau}$ . The  $L^p$ -space of persistence diagrams over the pair  $(X, A)$  is the set

$$\mathcal{D}_p(X, A) = \left\{ \sigma \in \mathcal{D}(X, A) : \sum_{x \in \sigma} \text{dist}(x, A)^p < \infty \right\}$$

endowed with the metric  $d_p$ .

## Theorem (Bubenik, Hartsock, '24; C., Galaz-García, Guijarro, Membrillo-Solís, '24)

Let  $X$  be a metric space,  $A \subset X$  a non-empty and closed subset, and  $p \in [1, \infty)$ . Then the following conditions hold:

- If  $X$  is complete then  $\mathcal{D}_p(X, A)$  is complete.
- If  $X$  is separable then  $\mathcal{D}_p(X, A)$  is separable.
- If  $X$  is proper, then optimal partial bijections exist.
- If  $X$  is geodesic and proper then  $\mathcal{D}_p(X, A)$  is geodesic too. Furthermore, if  $p \in (1, \infty)$ , every geodesic in  $\mathcal{D}_p(X, A)$  is given by an interpolation along geodesics of  $X$ .
- If  $p \in (1, \infty)$  and  $X$  is non-branching, then  $\mathcal{D}_p(X, A)$  is non-branching.
- If  $X$  is a proper non-negatively curved Alexandrov space then  $\mathcal{D}_2(X, A)$  is also a non-negatively curved Alexandrov space.

## Theorem (Divol, Lacombe, '21; C., '24)

*If  $X$  is proper and  $p \in [1, \infty)$  then  $\mathcal{D}_p(X, A)$  isometrically embeds into  $\mathcal{M}_p(X, A)$ .*

## Definition

Let  $(E, \|\cdot\|_E)$  be a Banach space, where  $E \subset \mathbb{R}^{\mathbb{N}}$ . We say that  $(E, \|\cdot\|_E)$  is:

- *permutation invariant* if for any bijection  $\phi: \mathbb{N} \rightarrow \mathbb{N}$  and any  $u \in E$ ,  $u_\phi \in E$  and

$$\|u_\phi\|_E = \|u\|_E,$$

where  $u_\phi(n) = u(\phi(n))$ ;

- *normalised* if  $e_i \in E$  and  $\|e_i\|_E = 1$  for all  $i \in \mathbb{N}$ , where  $e_i$  is the  $i$ -th canonical sequence;
- a *Banach sequence ideal* if, whenever  $u \in \mathbb{R}^{\mathbb{N}}$ ,  $v \in V$  and  $|u| \leq |v|$ , it follows that  $u \in V$ , and if  $u, v \in E$  and  $|u| \leq |v|$  then  $\|u\|_E \leq \|v\|_E$ .
- *$\sigma$ -order continuous* if whenever  $\{u_m\} \subset E$  satisfies  $0 \leq u_{m+1} \leq u_m$  and  $u_m \rightarrow 0$  pointwise, it follows that  $u_m \rightarrow 0$  in  $E$ .
- *Banach functional sequential space* if it is a permutation invariant, normalised,  $\sigma$ -order continuous Banach sequence ideal.

# Banach functional norms and persistence diagrams

## Definition

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- *Banach functional sequential space* if it is a permutation invariant, normalised,  $\sigma$ -order continuous Banach sequence ideal.

## Example

$\ell_p$  with  $p \in [1, \infty)$ ;  $c_0$ ; Orlicz spaces of sequences.

## Theorem (C., Galaz-Fontes, Galaz-García (in progress))

Let  $(E, \|\cdot\|_E)$  be a Banach functional sequential space,  $X$  a metric space, and  $A \subset X$  a closed and non-empty subset. Then we can induce a (extended) metric on  $\mathcal{D}(X, A)$  given by

$$d_E(\sigma, \tau) = \inf_{\phi} \|(d(x, \phi(x)))_{x \in \tilde{\sigma}}\|_E$$

where the infimum runs over all bijections  $\phi: \tilde{\sigma} \rightarrow \tilde{\tau}$  between representatives of  $\sigma$  and  $\tau$ . Restricted to the appropriate set, this yields metric spaces  $\mathcal{D}_E(X, A)$ . Moreover, the following implications hold:

- If  $X$  is complete then  $\mathcal{D}_E(X, A)$  is also complete.
- If  $X$  is separable then  $\mathcal{D}_E(X, A)$  is also separable.
- If  $X$  is proper, then there exist optimal bijections in  $\mathcal{D}_E(X, A)$ .
- If  $X$  is proper and geodesic, then  $\mathcal{D}_E(X, A)$  is also geodesic.

Thank you for your attention!