On isometric embeddings of metric spaces

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Contents

Acknowledgments iii

Introducción y resultados obtenidos v

Introduction and statement of results xiii

1 Preliminaries 1
- 1.1 Geodesic measure spaces 1
- 1.2 Curvature on geodesic metric spaces 3
  - 1.2.1 Comparisons for distance functions 3
  - 1.2.2 Distance comparison for triangles 4
  - 1.2.3 Angle comparison for triangles 6
- 1.3 Some convexity considerations about the distance function of a metric space . . 7
- 1.4 Submersions and submetries 8
- 1.5 Optimal Transport and Wasserstein type spaces 9
  - 1.5.1 Wasserstein space 10
  - 1.5.2 Orlicz–Wasserstein space 12
  - 1.5.3 Space of Persistence Diagrams 12
- 1.6 Reach of a subset 14

2 Filling Radius and the L^∞ space of a Riemannian manifold 17
- 2.1 The Kuratowski embedding 17
- 2.2 Filling radius 18
- 2.3 Positiveness of the Filling Radius 22
- 2.4 Filling Radius bounds of Riemannian submersions 24
- 2.5 Filling radius of submetries 27
- 2.6 Intermediate Filling Radius 27

3 On the reach of isometric embeddings of metric spaces 31
- 3.1 Reach of the Kuratowski embedding 31
- 3.2 Reach of the Wasserstein space 33
  - 3.2.1 Null reach 33
  - 3.2.2 Infinite reach 37
  - 3.2.3 Projection map 39
- 3.3 Reach in the Orlicz–Wasserstein space 40
  - 3.3.1 Null reach 40
  - 3.3.2 Positive Reach 44

3.4 Reach of the Persistence Diagram Space 44

4 The Rival Coffee Shop Problem **47**
4.1 The Coffee Shop Problem 47
4.1.1 State of the art 49
4.1.2 Competition 51
4.2 Fixed competition 51
4.3 Non–fixed competition 55
4.3.1 Forbidden areas 55
4.3.2 Rival growth in terms of ours 56
4.4 Discussion about the constants in the difference of Dirac deltas 59

Conclusions **61**

Conclusiones **63**

Bibliography **73**

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Introducción y resultados obtenidos

Extraer y describir propiedades geométricas y topológicas de ciertos espacios métricos es, en muchas ocasiones, una tarea difícil y costosa. Por ello, usar técnicas que impliquen no enfrentarse directamente con estos objetos suele ser una estrategia inteligente. Una de las más usadas consiste en el encaje de estos espacios dentro de otros llamados *espacios ambiente*. A través de esta inclusión, no solo podemos llegar a concluir resultados sobre el espacio inicial sino, también, sobre el espacio ambiente.

Una de las características que más se intenta preservar a través de estos encajes es la de la distancia, es decir, se busca que esta se conserve dentro del nuevo espacio. Formalmente, se intenta trabajar con encajes *isométricos*: la función distancia del espacio ambiente restringida a la imagen del encaje de nuestro espacio inicial tiene que coincidir con la función distancia del espacio métrico original.

Son muchos los resultados en torno a este tipo de encajes y algunos han alcanzado bastante fama debido a su relevancia. Un buen ejemplo es el encaje de variedades suaves de Nash [72]. El estadounidense probó que toda variedad riemanniana suave puede ser encajada de modo isométrico dentro de un espacio euclídeo de dimensión suficientemente alta. Otro ejemplo con más de cien años e igualmente importante, fue el resultado dado por Hilbert [45] donde probaba que no había ninguna copia isométrica completa del espacio hiperbólico dentro de $\mathbb{R}^3$. Sin duda se podría proveer una lista interminable con resultados clave para distintas áreas donde este tipo de encajes juegan un papel fundamental y no haría más que mostrar la importancia que tiene obtener resultados relacionados con ellos.

En esta tesis doctoral vamos a estudiar dos encajes isométricos en particular: el encaje de Kuratowski y el encaje canónico dentro de espacios de tipo Wasserstein.

El encaje isométrico de Kuratowski ϕ es el encaje natural de un espacio métrico compacto X dentro del espacio $L^\infty(X)$, donde

$$L^\infty(X) = \left\{ f \colon X \to \mathbb{R} \colon ||f||_\infty = \sup_{x \in X} |f(x)| < \infty \right\},$$

producido por la siguiente aplicación

$$\begin{aligned} \phi \colon X &\to L^\infty(X) \\ x &\mapsto \operatorname{dist}_x(\cdot) := \operatorname{dist}_X(x, \cdot). \end{aligned}$$

Cuando X es una variedad riemanniana (M^n, g) cerrada –compacta y sin frontera–, este encaje se usa para calcular el Filling Radius, invariante definido por Gromov en [39]. Fred Wihelm propone en [93] entender intuitivamente el Filling Radius del siguiente modo: podemos considerar que una variedad riemanniana (M^n, g) orientable y cerrada acota un agujero $(n+1)$–dimensional; entonces el Filling Radius mediría el tamaño de dicho agujero.

Aunque la definición de Filling Radius es bastante natural y pareciera que esconde bastante información sobre nuestros espacios, solo se conocen unos pocos valores exactos de este invariante (ver Sección 2.1, además de las referencias [53, 54, 55]). Por este motivo, contribuciones en torno a calcular valores o aportar desigualdades de cara a acotar el valor del Filling Radius son de relativa importancia.

En esta tesis presentamos cotas tanto inferiores como superiores al Filling Radius. En primer lugar, probamos en [31] que el Filling Radius de una variedad cerrada siempre es positivo. Este resultado ya había aparecido con anterioridad [37], pero presentamos una nueva prueba:

Teorema 2.6. *Sea M una variedad riemanniana cerrada con radio de inyectividad* $\operatorname{inj} M$ *y curvatura seccional* $\sec \leq K$*, donde $K \geq 0$. Entonces*

$$\operatorname{FillRad}(M) \geq \frac{1}{4} \min \left\{ \operatorname{inj} M \,,\, \frac{\pi}{\sqrt{K}} \right\}, \tag{1}$$

donde $\pi/\sqrt{K}$ se considera ∞ cuando $K = 0$.

Gracias a este resultado obtenemos la cota deseada:

Corolario 2.6.1. *Sea (M, g) una variedad riemanniana cerrada, entonces*

$$\operatorname{FillRad}(M) \geq c_0 > 0.$$

La cota superior que obtenemos para el Filling Radius de una variedad cerrada M necesita de una submersión riemanniana –para más información sobre este tipo de aplicaciones, ver la Sección 1.4– entre M y otra variedad B:

Teorema 2.7. *Sea $\pi : M \to B$ una submersión riemanniana donde* $\dim M > \dim B$. *Entonces*

$$\operatorname{FillRad}(M) \leq \frac{1}{2} \max_{b \in B} \{ \operatorname{diam} \pi^{-1}(b) \}, \tag{2}$$

donde el diámetro de cada fibra es tomado respecto a la métrica extrínseca.

Una vez obtenido este resultado, hay varios corolarios cambiando un poco el tipo de aplicación entre espacios. En primer lugar, si relajamos un poco la restricción de que los espacios sean variedades riemannianas, podemos usar submetrías (una generalización métrica del concepto de submersiones riemannianas).

Corolario 2.7.2. *Sea $(X, \widehat{\operatorname{dist}_X})$ una variedad métrica (i.e, una variedad cerrada con una distancia definida), $(Y, \operatorname{dist}_Y)$ un espacio métrico y $\pi : X \to Y$ una submetría entre ellos. Entonces*

$$\operatorname{FillRad}(X) \leq \frac{1}{2} \max_{y \in Y} \{ \operatorname{diam} \pi^{-1}(y) \}.$$

También, en los Corolarios 2.7.1 y 2.7.3 se puede encontrar el mismo resultado pero con productos alabeados y foliaciones riemannianas singulares.

La definición de Filling Radius involucra la clase fundamental en homología de una variedad riemanniana. Siguiendo esa noción en grupos de homología de dimensiones menores, se puede replicar la definición de Filling Radius dando lugar a los k–Filling Radius intermedios. Ningún valor exacto se conoce de estos invariantes. En este sentido, todo lo que se ha aportado hasta la fecha son cotas. Tomando dos variedades cerradas y suponiendo una aplicación Lipschitz entre ellas, en [31] obtuvimos el siguiente resultado:

Teorema 2.9. *Sea $f\colon M^m \to N^n$ una aplicación Lipschitz entre variedades cerradas, tal que la función inducida $f_{k,*} : \mathrm{H}_k(M) \to \mathrm{H}_k(N)$ sea sobreyectiva. Entonces*

$$\mathrm{FillRad}_k(M) \geq C^{-1}\,\mathrm{FillRad}_k(N).$$

El *reach* de un subconjunto $A \subset X$ –definido por Federer en [32]– mide, localmente, el máximo radio de la bola centrada en ese punto cuyos puntos tienen una única proyección métrica en A. Es decir, de un modo informal, si encajamos un espacio dentro de otro ambiente, el reach mide cuán arrugada o estirada queda la imagen de dicho encaje. En la Sección 1.6 hemos proporcionado una recopilación de referencias bibliográficas con suficientes resultados relevantes involucrando el reach para aquel lector interesado en la materia.

Para terminar con el estudio del encaje de Kuratowski, en [31] calculamos el valor de su reach:

Teorema 3.1. *Sea M^n una variedad riemanniana compacta. Entonces, para todo $p \in M$ se tiene que*

$$\mathrm{reach}(p, M \subset L^\infty(M)) = 0.$$

El segundo encaje isométrico que estudiamos a lo largo de la tesis es el que se produce de modo natural entre espacios métricos y espacios de tipo Wasserstein. En particular, tres de ellos: el usual –conocido como p–espacio de Wasserstein–, el Orlicz–Wassertein y el espacio de diagramas de persistencia. Las definiciones, introducción y motivación para el uso de ese tipo de espacios se pueden encontrar en la Sección 1.5. Todos los resultados obtenidos involucrando el reach y espacios de tipo Wasserstein fueron realizados junto a Javier Casado y Jaime Santos–Rodríguez y pueden ser encontrados en [24].

En lo relativo al p–espacio de Wassertein –el espacio de medidas de probabilidad con soporte el espacio métrico y p–momento finito–, primero demostramos cómo todo espacio métrico geodésico tiene reach nulo en el 1–espacio de Wassertein:

Teorema 3.2. *Sea (X, dist) un espacio métrico geodésico y $W_1(X)$ su 1–espacio de Wasserstein. Entonces, para cada punto de acumulación $x \in X$, tenemos que $\mathrm{reach}(x, X \subset W_1(X)) = 0$. Concretamente, si X no es un espacio discreto, se cumple que $\mathrm{reach}(X \subset W_1(X)) = 0$.*

Al permitir $p > 1$, el escenario se vuelve más complejo, ya que no podemos replicar el resultado anterior con los mismos argumentos. En [24] probamos que $\mathrm{reach}(X \subset W_p(X)) = 0$ estaba relacionado con la existencia de más de una geodésica minimal entre dos puntos. La siguiente proposición es la clave para ello:

Proposición 3.2. *Sea (X, dist) un espacio métrico geodésico y $x, y \in X$ dos puntos con $x \neq y$. Consideremos la medida de probabilidad $\mu = \lambda\delta_x + (1-\lambda)\delta_y$, con $0 < \lambda < 1$. Entonces μ minimiza su p–distancia de Wasserstein con respecto a X exactamente en un punto por cada geodésica minimal que exista entre x e y.*

Usando la Proposición 3.2 como herramienta fundamental, obtuvimos el siguiente teorema para p–espacios de Wasserstein con $p > 1$:

Teorema 3.3. *Sea (X, dist) un espacio métrico geodésico y $x \in X$ un punto tal que existe otro $y \in X$ con la propiedad de que existen al menos dos geodésicas minimales distintas entre ellos. Entonces, para todo $p > 1$ se cumple que*

$$\mathrm{reach}(x, X \subset W_p(X)) = 0.$$

En particular, si existe un $x \in X$ que satisface esa propiedad, $\operatorname{reach}(X \subset W_p(X)) = 0$ *para todo* $p > 1$.

A raíz de este teorema surgieron los Corolarios 3.3.1, 3.3.2 y 3.3.3 que prueban que el $\operatorname{reach}(X \subset W_p(X)) = 0$ para variedades compactas, no simplemente conexas y espacios métricos propios geodésicos.

De un modo natural, la siguiente pregunta que intentamos responder fue la de la positividad del reach en los p–espacios de Wassertein. Fue respondida parcialmente gracias al siguiente teorema:

Teorema 3.5. *Sea* (X, dist) *un espacio métrico reflexivo. Las siguientes afirmaciones son ciertas:*

1. *Si X es estrictamente p–convexo para $p \in [1, \infty)$ o uniformemente ∞–convexo si $p = \infty$, entonces*
$$\operatorname{reach}(X \subset W_r(X)) = \infty, \text{ para } r > 1. \tag{3}$$
2. *Si X es Busemann, estrictamente p–convexo para algún $p \in [1, \infty]$ y uniformemente q–convexo para algún $q \in [1, \infty]$, entonces*
$$\operatorname{reach}(X \subset W_r(X)) = \infty, \text{ para } r > 1. \tag{4}$$

Al igual que en el caso nulo, este último teorema se puede aplicar a espacios conocidos, como los CAT (0) (Corolario 3.5.1).

El estudio del reach, como explicamos a lo largo de esta tesis, está intimamente relacionado con la existencia de entornos tubulares en los cuales todo punto tenga una proyección métrica única sobre el subconjunto al que se le pretende calcular el reach. Siempre que exista un $\epsilon > 0$ tal que $\operatorname{reach}(x, X \subset W_p(X)) \geq \epsilon$ para todo $x \in X$, podemos definir la función $proj_p \colon W_p(X) \to X$ en el ϵ–entorno de $X \subset W_p(X)$, que envíe cada medida μ a su p–baricentro –término usado en la literatura para referirse a la proyección métrica en terminos de distancia. En el caso de $p = 2$ pudimos probar el siguiente resultado que trata sobre la regularidad de la función $proj_2$ e involucra a espacios cuyo reach es infinito, por tanto, tiene la función $proj_2$ definida en todo $W_2(X)$:

Teorema 3.6. *Sea* $(X, \|\cdot\|)$ *un espacio de Banach reflexivo con una norma estrictamente convexa y que satisface la propiedad **B** [57, Sección 4.3]. Entonces* $proj_2 \colon W_2(X) \to X$ *es una submetría.*

Los espacios de Orlicz–Wasserstein fueron definidos por Sturm en [82] de cara a establecer un contexto más general a los ya existentes p–espacios de Wasserstein. Estos espacios involucran dos funciones –una convexa φ y una cóncava ψ– en su construcción que, si son elegidas de un modo particular, hace que el espacio Orlicz–Wasserstein obtenido coincida con el clásico p–Wasserstein. Debido a ello, en [24], hicimos el mismo estudio del reach que con los canónicos, ya que también se pueden conseguir encajes isométricos de espacios metricos dentro de ellos.

Empezamos analizando cuándo el reach se anulaba y obtuvimos una proposición similar a la Proposición 3.2:

Proposición 3.3. *Sea X un espacio métrico geodésico y x, $y \in X$ tal que $x \neq y$. Definimos la siguiente medida de probabilidad $\mu = \lambda\delta_x + (1-\lambda)\delta_y$, para $0 < \lambda < 1$. Entonces, las siguientes afirmaciones se cumplen:*

1. *μ minimiza su ϑ–distancia de Wasserstein a X en una geodésica minimal entre x e y.*

2. *Si λ está cerca de uno y existe una constante $c > 1$ tal que $\varphi^{-1}(t) < t$ para todo $t > c$, entonces el mínimo se alcanzará en el interior de cada geodésica minimal.*

Por tanto, siguiendo la misma estructura que en el caso anterior, también somos capaces de obtener un teorema relacionando la existencia de más de una geodésica minimal con la nulidad del reach:

Teorema 3.7. *Sea X un espacio métrico geodésico y $x \in X$ un punto tal que cumple lo siguiente: existe al menos un punto $y \in X$ tal que está unido a x por al menos dos geodésicas minimales. Supongamos que X está isométricamente encajado en un espacio de Orlicz–Wasserstein $W_\vartheta(X)$. Entonces, para cada φ tal que $\varphi(t_0) \neq t_0$ para algún $t_0 > 1$ se tiene que*

$$\mathrm{reach}(x, X \subset W_\vartheta(X)) = 0.$$

En particular, si existe un $x \in X$ con la anterior propiedad citada, $\mathrm{reach}(X \subset W_\vartheta(X)) = 0$ *para cada $p > 1$. Además, en variedades compactas y no simplemente conexas,*

$$\mathrm{reach}(x, X \subset W_\vartheta(X)) = 0$$

para todo $x \in X$.

Del mismo modo que con los espacios de Wassertein canónicos, también obtuvimos resultados en relación a la positividad del reach para encajes isométricos dentro de los Orlicz–Wasserstein:

Teorema 3.8. *Sea (X, dist) un espacio CAT (0) reflexivo. Supongamos que φ es convexa y puede ser escrita como $\varphi(r) = \psi(r^p)$, donde ψ es otra función convexa y $p > 1$. Entonces*

$$\mathrm{reach}(X \subset W_\vartheta(X)) = \infty, \tag{5}$$

donde $\psi \equiv \mathrm{Id}$ y $\varphi(1) = 1$.

El último espacio de tipo Wasserstein que decidimos estudiar fue el espacio de diagramas de persistencia equipado con una distancia ∞–Wasserstein conocida como la *distancia de Bottleneck*. Los diagramas de persistencia son la clave del *Análisis de Datos Topológico* –conocido por TDA debido a sus iniciales en inglés– ya que capturan toda la información sobre el resultado del análisis aplicado a una nube de puntos particular. En [18], Bubenik y Wagner construyen de modo explícito un encaje isométrico para espacios métricos separados y acotados dentro del espacio de diagramas de persistencia. Debido a su importancia en la actualidad, decidimos estudiar este espacio y, en particular, el reach de este tipo de encajes:

Teorema 3.9. *Sea (X, dist) un espacio métrico separado y acotado y $(\mathrm{Dgm}_\infty, w_\infty)$ el espacio de diagramas de persistencia dotado con la distancia de bottleneck. Si $x \in X$ es un punto de acumulación, entonces*

$$\mathrm{reach}(x, X \subset \mathrm{Dgm}_\infty) = 0.$$

En particular, si X no es discreto, $\mathrm{reach}(X \subset \mathrm{Dgm}_\infty) = 0$.

Desde finales del siglo XX, tanto el transporte óptimo como las técnicas derivadas de los espacios de tipo Wasserstein [80, 92] han ido creando nuevas vías para resolver problemas clásicos. Un buen ejemplo es el que se desarrolla en el Capítulo 4 de la presente tesis, el cual estudia el *Problema de las cafeterías* con una nueva variación. Su enunciado original pregunta lo siguiente: cómo colocar N establecimientos de forma óptima en una cierta región X. Este problema se deriva de una amplia tradición de problemas de localización y transporte que, a través de varias enunciaciones, intentan aproximar puntos en un espacio a la distribución uniforme en la misma. En este caso en particular, además, en [23] introducimos nueva hipótesis a este famoso problema de colocación de establecimientos: competencia.

Steinerberger y Brown [15, 16, 81] decidieron trabajar este problema bajo la óptica del transporte óptimo y usar el encaje isométrico de la región en cuestión dentro de su 2–espacio de Wasserstein con el fin de poder usar la distancia allí definida y acotar la distancia entre las cafeterías –representada como una suma de deltas de dirac normalizadas– y la medida de Lebesgue de la región con masa 1, debido a que estamos trabajando con medidas de probabilidad.

En [23], decidimos seguir con con ese mismo planteamiento pero añadiendo la competencia como una resta de deltas de Dirac a la medida inicial formada por nuestros comercios –en la Sección 4.4 se puede encontrar una discusión sobre el por qué de la elección de la nueva medida. A la hora de estudiar la casuística que producía esta nueva configuración decidimos dividir los resultados en dos grandes familias: la competencia es entre un número fijo de establecimientos o, bien, tiene un comportamiento dinámico.

El resultado principal en el caso de que la competencia esté fija, nos permite comprobar que, si ellos no modifican el número de establecimientos y nosotros vamos creciendo, nuestra colocación terminará ganando a la suya cualesquiera que sea esta:

Teorema 4.5. *Sea X una variedad d–dimensional suave, compacta y sin frontera donde $g \geq 3$, $G\colon X \times X \to \mathbb{R} \cup \{\infty\}$ la función de Green del laplaciano normalizada con media 0 sobre la variedad y N_1, $N_2 > 0$. Entonces, para cada conjunto de puntos distintos $\{x_1, \dots, x_{N_1}\}$ y $\{y_1, \dots, y_{N_2}\}$ se tiene que*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) \lesssim_{X,N_2} \frac{1}{(N_1+N_2)^{1/d}} + \frac{1}{N_1+N_2}\left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2},$$

donde $\mathbf{W}_1^{1,1}$ denota la distancia de Wasserstein con signo definida en [67] y en la Sección 1.5.1, $z_i = x_i$ de $i = 1$ a N_1 y $z_i = y_{i-N_1}$ de $i = N_1 + 1$ a $N_1 + N_2$.

En la parte derecha de la desigualdad del anterior teorema aparece como sumando la función de Green –explicada con detalle en la Sección 4.2. Al igual que en los artículos de Steinerberger y Brown, si las cafeterías de nuestra propiedad son colocadas convenientemente, el sumando desaparece de la desigualdad:

Teorema 4.6. *Sea z_n una sucesión construida como en (4.5) sobre una variedad d–dimensional compacta con $d \geq 3$ y $\{x_1, \dots, x_{N_1}\} \subset \{z_i\}_{i=1}^{N_1+N_2}$ y $\{y_1, \dots, y_{N_2}\} \subset \{z_i\}_{i=1}^{N_1+N_2}$ tal que $x_i \neq y_j$ para todos i, j. Entonces*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) \lesssim_{X,N_2} \frac{1}{(N_1+N_2)^{1/d}}.$$

A la hora de trabajar con el caso dinámico, en primer lugar, a través de la Proposición 4.2 y el Corolario 4.2.1 probamos que, a través del establecimiento de zonas de medida positiva donde no se nos permite abrir comercios pero el rival sí puede, la competencia siempre ganaría.

Finalmente, quitando la restricción de las zonas prohibidas, probamos los dos últimos teoremas del capítulo para diferentes tipos de crecimiento de la competencia:

Teorema 4.7. *Sea* $\mu_N = \left(\sum_{i=1}^{N} \delta_{x_i} - \sum_{j=1}^{f(N)} \delta_{y_j}\right)$. *Si* $f(N) \geq f(N-1)+2$, *entonces, para* N_0 *suficientemente grande, los establecimientos de la competencia ganarán para todo* $N \geq N_0$, *es decir,*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}\mu_N, dx\right) > \mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}(-\mu_N), dx\right). \tag{6}$$

Theorem 4.8. *Sea* $\mu_N = \left(\sum_{i=1}^{N} \delta_{x_i} - \sum_{j=1}^{N+K} \delta_{y_j}\right)$, *y* $N_0 > 0$. *Entonces, existen valores de* K *tal que la competencia tendrá una estrategia ganadora para todo* $N \leq N_0$, *es decir,*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{2N+K}\mu_N, dx\right) > \mathbf{W}_1^{1,1}\left(\frac{1}{2N+K}(-\mu_N), dx\right). \tag{7}$$

La tesis está organizada del siguiente modo:

El Capítulo 1 está dedicado a asentar las bases y preliminares necesarios para la comprensión del trabajo: comenzamos recordando la definición de un espacio métrico geodésico junto con el concepto de curvatura en el marco de espacios métricos (Secciones 1.1 y 1.2). Dedicamos la Sección 1.3 a definiciones sobre ciertos tipos de convexidad aplicados a la función distancia. Estos serán usados posteriormente para probar algunos teoremas del Capítulo 3. En la Sección 1.4 recordamos las definiciones de submersión riemanniana y submetrías –una generalización de la submersión en el caso de espacios métricos en vez de variedades– así como algunos resultados relevantes de este último tipo de funciones. Como final del capítulo, en la Sección 1.5 introducimos el problema del transporte óptimo como paso previo a presentar los espacios de Wasserstein con los que vamos a trabajar a lo largo de la tesis y en la Sección 1.6 definimos el reach –en el sentido de Federer– a la vez que aportamos algunos resultados relevantes para dar contexto histórico.

El Capítulo 2 está dedicado al encaje isométrico de una variedad cerrada M en su espacio $L^\infty(M)$ y el invariante Filling Radius definido por Gromov en [39]. En la Sección 2.3 probamos en detalle la positividad del Filling Radius. En las Secciones 2.4 y 2.5 presentamos las cotas superiores para el Filling Radius en presencia de submersiones riemannianas y submetrías, además de para productos alabeados y foliaciones riemannianas singulares. Y, por último, en la Sección 2.6 replicamos algunos resultados para el Filling Radius intermedio así como una cota en presencia de aplicaciones Lipschitz entre variedades.

El Capítulo 3 lo dedicamos a todos los resultados que involucran el reach. En primer lugar, en la Sección 3.1 calcularemos el reach del encaje de Kuratowski. En el resto de Secciones 3.2, 3.3 y 3.4, se calcularán los reach para los espacios de tipo Wasserstein que se presentan en los preliminares.

Para terminar, en el Capítulo 4 abordaremos el problema de las cafeterías con competición –el cual explicaremos en detalle en la Sección 4.1– dividiéndolo en dos casos: el primero en la Sección 4.2 cuando la competencia es fija y el segundo en la Sección 4.3 cuando la competencia es dinámica. Finalmente, en la Sección 4.4, presentaremos una pequeña discusión sobre la elección de la medida de probabilidad usada durante el estudio del problema.

Introduction and statement of results

Extracting and describing geometric and topological properties of certain metric spaces is often a challenging and costly task. Therefore, employing techniques that do not directly engage with these objects is often a wise strategy. One of the most commonly used approaches involves embedding these spaces into other spaces known as *ambient spaces*. Through this embedding, we can not only draw conclusions about the initial space but also about the ambient one.

One of the key characteristics that is often sought to be preserved through these embeddings is distance; in other words, the aim is to maintain distance within the new space. Formally, the goal is to work with *isometric embeddings*: the distance function of the ambient space restricted to the image of the embedding of our initial space must coincide with the distance function of the original metric space.

There are numerous results concerning such embeddings, and some have gained considerable success due to their significance. A prime example is the embedding of smooth Riemannian manifolds by Nash [72]. He proved that every smooth Riemannian manifold can be isometrically embedded into a sufficiently high-dimensional Euclidean space. Another example, equally important and dating back over a century, is the result established by Hilbert [45], where he stated that there is no complete isometric copy of hyperbolic space within $\mathbb{R}^3$. Undoubtedly, one could provide an endless list of key results in various fields where such embeddings play a fundamental role, highlighting the importance of obtaining results related to them.

In this dissertation, we will study two specific isometric embeddings: the Kuratowski embedding and the canonical embedding into Wasserstein-type spaces.

The Kuratowski isometric embedding ϕ is the natural embedding of a compact metric space X into $L^\infty(X)$, where

$$L^\infty(X) = \left\{ f\colon X \to \mathbb{R}\colon ||f||_\infty = \sup_{x\in X} |f(x)| < \infty \right\},$$

defined by the following map

$$\begin{aligned} \phi\colon X &\to L^\infty(X) \\ x &\mapsto \operatorname{dist}_x(\cdot) := \operatorname{dist}_X(x,\cdot). \end{aligned}$$

When X is a closed (compact and without boundary) Riemannian manifold (M^n, g), this embedding is used to calculate the Filling Radius, an invariant defined by Gromov in [39]. Fred Wilhelm proposes in [93] an intuitive understanding of the Filling Radius as follows: we can consider that a closed orientable Riemannian manifold (M^n, g) bounds a $(n+1)$–dimensional hole; then the Filling Radius measures the size of this hole.

Although the definition of the Filling Radius is quite natural and seems to contain valuable information about our spaces, only a few exact values of this invariant are known (see Section 2.1, as well as references [53, 54, 55]). For this reason, contributions related to calculating values or providing inequalities to bound the Filling Radius are of relative importance.

In this thesis, we present both lower and upper bounds on the Filling Radius. Firstly, we prove in [31] that the Filling Radius of a closed manifold is always positive. This result had already appeared in some works, but we present here a new proof:

Theorem 2.6. *Let M be a closed Riemannian manifold with injectivity radius $\operatorname{inj} M$ and with sectional curvature $\sec \leq K$, where $K \geq 0$. Then*

$$\operatorname{FillRad}(M) \geq \frac{1}{4} \min \left\{ \operatorname{inj} M \,,\, \frac{\pi}{\sqrt{K}} \right\}, \tag{8}$$

where $\pi/\sqrt{K}$ is understood as ∞ whenever $K = 0$.

With this result we obtain the desired bound:

Corollary 2.6.1. *Let (M, g) be a closed Riemannian manifold, then*

$$\operatorname{FillRad}(M) \geq c_o > 0.$$

The upper bound we obtain for the Filling Radius of a closed manifold M requires a Riemannian submersion (for more information on this type of function, see Section 1.4) between M and another manifold B known as *base*:

Theorem 2.7. *Let $\pi : M \to B$ be a Riemannian submersion with $\dim M > \dim B$. Then*

$$\operatorname{FillRad}(M) \leq \frac{1}{2} \max_{b \in B} \{\operatorname{diam} \pi^{-1}(b)\}, \tag{9}$$

where the diameter of each fiber is considered in the extrinsic metric.

Once this result is established, there are several corollaries that arise when slightly modifying the type of function between spaces. Firstly, if we relax the restriction that the spaces are Riemannian manifolds, we can use submetries (a metric generalization of the concept of Riemannian submersions).

Corollary 2.7.2. *Let $(X, \widehat{\operatorname{dist}}_X)$ be a metric manifold (i.e, a closed manifold with a distance), $(Y, \operatorname{dist}_Y)$ a metric space and $\pi : X \to Y$ a submetry between them. Thus*

$$\operatorname{FillRad}(X) \leq \frac{1}{2} \max_{y \in Y} \{\operatorname{diam} \pi^{-1}(y)\}.$$

Also, in Corollaries 2.7.1 and 2.7.3, the same result can be found, but with warped products and singular Riemannian foliations.

The definition of the Filling Radius involves the fundamental class in the n–homology group of a Riemannian manifold. Following a similar notion in lower-dimensional homology groups, we can define the intermediate k-Filling Radius. No exact values are known for these invariants. In this regard, all that has been contributed to date are bounds. Considering two closed manifolds and assuming a Lipschitz map between them, in [31] we obtained the following result:

Theorem 2.9. *Let $f : M^m \to N^n$ be a Lipschitz map between closed manifolds such that the induced map $f_{k,*} : \mathrm{H}_k(M) \to \mathrm{H}_k(N)$ is onto. Then*

$$\mathrm{FillRad}_k(M) \geq C^{-1}\, \mathrm{FillRad}_k(N).$$

The *reach* of a subset $A \subset X$ (defined by Federer in [32]) measures, locally, the maximum radius of a ball centered at a point in which its points have a unique metric projection onto A. In other words, in an informal sense, if we embed one space in an ambient one, the reach measures how wrinkled or stretched the image of this embedding becomes. In Section 1.6, we have provided a compilation of bibliographic references with sufficient relevant results involving the reach for readers interested in the subject.

To conclude the study of the Kuratowski embedding, in [31], we calculate the reach of that embedding:

Theorem 3.1. *Let M^n be a compact Riemannian manifold. For every $p \in M$,*

$$\mathrm{reach}(p, M \subset L^\infty(M)) = 0.$$

The other isometric embedding that we study throughout this thesis naturally occurs between metric spaces and Wasserstein-type spaces, specifically three of them: the standard one, known as the p-Wasserstein space, the Orlicz–Wasserstein space, and the space of persistence diagrams. Definitions, introductions, and motivations for the use of these types of spaces can be found in Section 1.5. All the results obtained concerning the reach and Wasserstein-type spaces were produced in collaboration with Javier Casado and Jaime Santos-Rodríguez and can be found in [24].

Regarding the p–Wasserstein space, which is the space of probability measures supported on the metric space with finite p–moments, we first proved how every geodesic metric space has zero reach in the 1-Wasserstein space:

Theorem 3.2. *Let (X, dist) be a geodesic metric space, and consider its 1–Wasserstein space, $W_1(X)$. Then, for every accumulation point $x \in X$,* $\mathrm{reach}(x, X \subset W_1(X)) = 0$. *In particular, if X is not discrete,* $\mathrm{reach}(X \subset W_1(X)) = 0$.

When allowing $p > 1$, the scenario becomes more complex, as we cannot replicate the previous result with the same arguments. In [24], we demonstrated that $\mathrm{reach}(X \subset W_p(X)) = 0$ was related to the existence of more than one minimal geodesic between two points. The following proposition is the key to this:

Proposition 3.2. *Let (X, dist) be a geodesic metric space, and $x, y \in X$ two points with $x \neq y$. Consider the probability measure $\mu = \lambda\delta_x + (1-\lambda)\delta_y$, for $0 < \lambda < 1$. Then μ minimizes its p–Wasserstein distance to X exactly once for every minimizing geodesic between x and y.*

Using Proposition 3.2 as a fundamental tool, we obtained the following theorem for p-Wasserstein spaces with $p > 1$:

Theorem 3.3. *Let X be a geodesic metric space, and $x \in X$ a point such that there exists another $y \in X$ with the property that there exist at least two different minimizing geodesics from x to y. Then, for every $p > 1$,*

$$\mathrm{reach}(x, X \subset W_p(X)) = 0.$$

In particular, if there exists a point $x \in X$ satisfying that property, $\mathrm{reach}(X \subset W_p(X)) = 0$ *for every $p > 1$.*

As a result of this theorem, the following Corollaries 3.3.1, 3.3.2, and 3.3.3 emerged, which prove that $\mathrm{reach}(X \subset W_p(X)) = 0$ for compact manifolds, non-simply connected spaces, and proper geodesic metric spaces.

Naturally, the next question we attempted to answer was the positivity of reach in p-Wasserstein spaces. Partially, it was addressed with the following theorem:

Theorem 3.5. *Let (X, dist) be a reflexive metric space. Then the following assertions hold:*

1. *If X is strictly p–convex for $p \in [1, \infty)$ or uniformly ∞–convex if $p = \infty$, then*

$$\mathrm{reach}(X \subset W_r(X)) = \infty, \text{ for } r > 1. \tag{10}$$

2. *If X is Busemann, strictly p–convex for some $p \in [1, \infty]$ and uniformly q–convex for some $q \in [1, \infty]$, then*

$$\mathrm{reach}(X \subset W_r(X)) = \infty, \text{ for } r > 1. \tag{11}$$

Just like in the null case, this last theorem can be applied to spaces known as CAT(0) spaces (Corollary 3.5.1).

The study of reach, as we have explained throughout this thesis, is intimately related to the existence of tubular neighborhoods in which every point has a unique metric projection onto the subset for which we intend to calculate the reach. Whenever there exists an $\epsilon > 0$ such that $\mathrm{reach}(x, X \subset W_p(X)) \geq \epsilon$ for all $x \in X$, we can define the function $proj_p \colon W_p(X) \to X$ within the ϵ-neighborhood of $X \subset W_p(X)$, which maps each measure μ to its p-barycenter (a term used in the literature to refer to metric projection in terms of distance). In the case of $p = 2$, we were able to prove the following result concerning the regularity of the function $proj_2$, which involves spaces with infinite reach and, therefore, has the function $proj_2$ defined throughout $W_2(X)$:

Theorem 3.6. *Let $(X, \|\cdot\|)$ be a reflexive Banach space equipped with a strictly convex norm and satisfying property **B**. Then $proj_2$ is a submetry.*

Orlicz–Wasserstein spaces were defined by Sturm in [82] with the aim of establishing a more general framework within the already existing p–Wasserstein spaces. These spaces involve two functions (one convex φ and one concave ψ) in their construction, which, if chosen in a particular way, makes the resulting Orlicz–Wasserstein space coincide with the classical p–Wasserstein space. Because of this, in [24], we conducted the same study of the reach as with the canonical ones, as it is also possible to achieve isometric embeddings of metric spaces into them.

We began by analyzing when the reach became zero and obtained a proposition similar to Proposition 3.2:

Proposition 3.3. *Let X be a geodesic metric space, and let $x, y \in X$ be two points with $x \neq y$. Consider the probability measure $\mu = \lambda\delta_x + (1 - \lambda)\delta_y$, for $0 < \lambda < 1$. Then, the following assertions hold:*

1. *μ can only minimize its ϑ–Wasserstein distance to X inside a minimizing geodesic between x and y.*

2. *If λ is close to one, and there exists a constant $c > 1$ such that $\varphi^{-1}(t) < t$ for every $t > c$, then the minimum will be attained inside the interior of each geodesic.*

Therefore, following the same structure as in the previous case, we are also able to obtain a theorem relating the existence of more than one minimal geodesic to the nullity of the reach:

Theorem 3.7. *Let X be a geodesic metric space, and $x \in X$ a point such that there exists another $y \in X$ with the property that there exists at least two different minimizing geodesics from x to y. Suppose X is isometrically embedded into an Orlicz-Wasserstein space $W_\vartheta(X)$. Then, for every φ such that $\varphi(t_0) \neq t_0$ for some $t_0 > 1$,*

$$\operatorname{reach}(x, X \subset W_\vartheta(X)) = 0.$$

In particular, if there exists a point $x \in X$ satisfying that property, $\operatorname{reach}(X \subset W_\vartheta(X)) = 0$ *for every $p > 1$. Also, in compact manifolds and non-simply connected manifolds,*

$$\operatorname{reach}(x, X \subset W_\vartheta(X)) = 0$$

for every $x \in X$.

Just like with the canonical Wasserstein spaces, we also obtained results regarding the positivity of reach for isometric embeddings within Orlicz-Wasserstein spaces:

Theorem 3.8. *Let (X, dist) be a reflexive $\operatorname{CAT}(0)$–space. Suppose φ is a convex function which can be expressed as $\varphi(r) = \psi(r^p)$, where ψ is another convex function and $p > 1$. Then*

$$\operatorname{reach}(X \subset W_\vartheta(X)) = \infty, \tag{12}$$

where $\psi \equiv \operatorname{Id}$ and $\varphi(1) = 1$.

The last type of Wasserstein space we decided to study was the space of persistence diagrams equipped with a ∞–Wasserstein distance, known as the *Bottleneck distance*. Persistence diagrams are key to *Topological Data Analysis*, commonly known as TDA, as they capture all the information about the result of the analysis applied to a specific point cloud. In [18], Bubenik and Wagner construct an explicit isometric embedding for separated and bounded metric spaces into the space of persistence diagrams. Because of its current significance, we decided to study this space and, in particular, the reach of these types of embeddings:

Theorem 3.9. *Let (X, dist) be a separable, bounded metric space and $(\operatorname{Dgm}_\infty, w_\infty)$ the space of persistence diagrams with the bottleneck distance. If $x \in X$ is an accumulation point, then*

$$\operatorname{reach}(x, X \subset \operatorname{Dgm}_\infty) = 0.$$

In particular, if X is not discrete, $\operatorname{reach}(X \subset \operatorname{Dgm}_\infty) = 0$.

Since the late 20th century, both optimal transport and techniques derived from Wasserstein-type spaces [80, 92] have been paving the way for solving classical problems. A good example of this can be found in Chapter 4 of this thesis, which explores the *Coffee Shop Problem* with a new variation. The original statement of this problem asks the following: how to optimally place N establishments within a certain region X. This problem is derived from a long tradition of location and transportation problems, which, through various formulations, attempt to approximate points in a space to a uniform distribution within the same space. In this particular case, we also introduce new assumptions to this well-known establishment placement problem: competition, as detailed in [23].

Steinerberger and Brown [15, 16, 81] decided to approach this problem from the perspective of optimal transport and use the isometric embedding of the relevant region X into their 2–Wasserstein space to be able to use the distance defined there and bound the distance between coffee shops (represented as a sum of normalized Dirac deltas) against the Lebesgue measure of the region with mass 1, since we are dealing with probability measures.

In [23], we decided to continue with the same approach but added competition as a subtraction of Dirac deltas from the initial measure formed by our shops (Section 4.4 provides a discussion on why we chose the new measure). When studying the scenarios that this new configuration produced, we decided to divide the results into two major categories: competition being a fixed number of establishments or having dynamic behavior.

The main result in the case where competition is fixed allows us to verify that if they do not change the number of establishments and we continue to grow, our placement will eventually outperform theirs, whatever it may be:

Theorem 4.5. *Let X be a smooth, compact d–dimensional manifold without boundary, $d \geq 3$, $G : X \times X \to \mathbb{R} \cup \{\infty\}$ denote the Green's function of the Laplacian normalized to have average value 0 over the manifold and $N_1, N_2 > 0$. Then for any distinct sets of points $\{x_1, \dots, x_{N_1}\}$ and $\{y_1, \dots, y_{N_2}\}$ we obtain*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) \lesssim_{X,N_2} \frac{1}{(N_1+N_2)^{1/d}} + \frac{1}{N_1+N_2}\left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2},$$

where $\mathbf{W}_1^{1,1}$ is the signed Wasserstein distance defined in [67] and in Section 1.5.1 and $z_i = x_i$ from $i = 1$ to N_1 and $z_i = y_{i-N_1}$ for $i = N_1 + 1$ to $N_1 + N_2$.

On the right-hand side of the inequality in the previous theorem, the Green's function appears as an additive term (explained in detail in Section 4.2). Just like in the articles by Steinerberger and Brown, if our coffee shops are strategically placed, this term disappears from the inequality:

Theorem 4.6. *Let z_n be a sequence obtained in the previous way on a d–dimensional compact manifold with $d \geq 3$ and $\{x_1, \dots, x_{N_1}\} \subset \{z_i\}_{i=1}^{N_1+N_2}$ and $\{y_1, \dots, y_{N_2}\} \subset \{z_i\}_{i=1}^{N_1+N_2}$ such that $x_i \neq y_j$ for all i, j. Then*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) \lesssim_{X,N_2} \frac{1}{(N_1+N_2)^{1/d}}.$$

When dealing with the dynamic case, first, through Proposition 4.2 and Corollary 4.2.1, we demonstrated that by establishing regions of positive measure where we are not allowed to open shops but the competitor can, the competition would always win.

Finally, removing the restriction of forbidden zones, we proved the last two theorems of the chapter for different types of growth of the competition:

Theorem 4.7. *Let* $\mu_N = \left(\sum_{i=1}^{N} \delta_{x_i} - \sum_{j=1}^{f(N)} \delta_{y_j}\right)$. *If* $f(N) \geq f(N-1) + 2$, *then, for* N_0 *big enough, the rival shops will win for all* $N \geq N_0$, *i.e.*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N + f(N)}\mu_N, dx\right) > \mathbf{W}_1^{1,1}\left(\frac{1}{N + f(N)}(-\mu_N), dx\right). \tag{13}$$

Theorem 4.8. *Let* $\mu_N = \left(\sum_{i=1}^{N} \delta_{x_i} - \sum_{j=1}^{N+K} \delta_{y_j}\right)$, *and* $N_0 > 0$. *Then, there exist values of* K *such that the rival shops will have a winning strategy for all* $N \leq N_0$, *i.e.*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{2N + K}\mu_N, dx\right) > \mathbf{W}_1^{1,1}\left(\frac{1}{2N + K}(-\mu_N), dx\right). \tag{14}$$

The thesis is organized as follows:

Chapter 1 is dedicated to laying down the necessary foundations and preliminaries for understanding the dissertation. We begin by recalling the definition of a geodesic metric space along with the concept of curvature within the framework of metric spaces (Sections 1.1 and 1.2). To conclude the discussion of basic concepts related to metric spaces, Section 1.3 provides definitions concerning certain types of convexity applied to the distance function. These definitions will be used later to prove some theorems in Chapter 3. In Section 1.4, we revisit the definitions of Riemannian submersion and submetry (a generalization of submersion in the case of metric spaces instead of manifolds) as well as some relevant results related to the latter type of functions. Finally, Section 1.5 introduces the problem of optimal transport as a precursor to presenting the Wasserstein spaces that we will work with throughout the thesis and in Section 1.6, we define reach (in the sense of Federer) while providing some relevant historical context.

Chapter 2 is dedicated to the isometric embedding of a closed manifold M into its space $L^\infty(M)$ and the Filling Radius invariant defined by Gromov in [39]. In Section 2.3, we thoroughly prove the positivity of the Filling Radius. In Sections 2.4 and 2.5, we present upper bounds for the Filling Radius in the presence of Riemannian submersions and submetries, as well as for warped products and singular Riemannian foliations. Finally, in Section 2.6, we replicate some results for the Intermediate Filling Radius and provide an upper bound in the presence of Lipschitz maps between manifolds.

Chapter 3 is devoted to all the results involving reach. First, in Section 3.1, we calculate the reach of the Kuratowski embedding. In the subsequent sections, 3.2, 3.3, and 3.4, we compute the reach for the Wasserstein type spaces presented in the preliminaries.

To conclude, in Chapter 4, we address the Coffee Shop Problem with competition, which we explain in detail in Section 4.1. We divide it into two cases: the first, in Section 4.2, when competition is fixed, and the second, in Section 4.3, when competition is dynamic. Finally, in Section 4.4, we provide a brief discussion on the choice of the probability measure used during the study of the problem.

Chapter 1

Preliminaries

In this chapter, we will establish the foundational framework required for a precise understanding of isometric embeddings. This will need an initial discussion concerning metric spaces and the essential constraints imposed upon them, including constraints on distance functions and the introduction of a curvature concept within metric spaces. Subsequently, we will introduce two distinct families of mappings applicable to Riemannian manifolds and, in a more general context, metric spaces. These mappings, *Riemannian submersions* and *submetries*, will appear in key results in subsequent chapters.

Many theorems of the thesis involve the use of Wasserstein distance, Wasserstein spaces or similar notions, so we spend on these preliminaries a whole section introducing all those concepts. Finally, we present the *reach* of a subset and some of the most relevant results involving it.

1.1 Geodesic measure spaces

We will begin with a complete and separable metric space (X, dist_X). The first restriction we will impose concerns the existence of minimizing (in terms of distance) curves between two points. Formally, the definition is as follows:

Definition 1 (Geodesic metric space)**.** Let (X, dist_X) be a metric space and $x, y \in X$. A curve $\gamma\colon [0,1] \to X$ will be called a *geodesic* between x and y if $\gamma(0) = x$, $\gamma(1) = y$ and

$$\mathrm{dist}_X(\gamma(t), \gamma(s)) = |t - s| \, \mathrm{dist}_X(x, y).$$

The set of all the geodesics of X will be denoted as $\mathrm{Geo}(X)$.

Example 1. An example of geodesic spaces are Riemannian manifolds with the Riemannian distance.

In spite of the existence of geodesics, these could not be unique between any two points or, either, we could have a branching phenomena in the space:

Definition 2 (Non-branching space)**.** A geodesic space (X, dist_X) will be called *non-branching* if the map

$$\begin{aligned} \mathrm{Geo}(X) &\to X \times X \\ \gamma &\mapsto (\gamma(0), \gamma(t)) \end{aligned}$$

is injective for all $t \in (0,1)$.

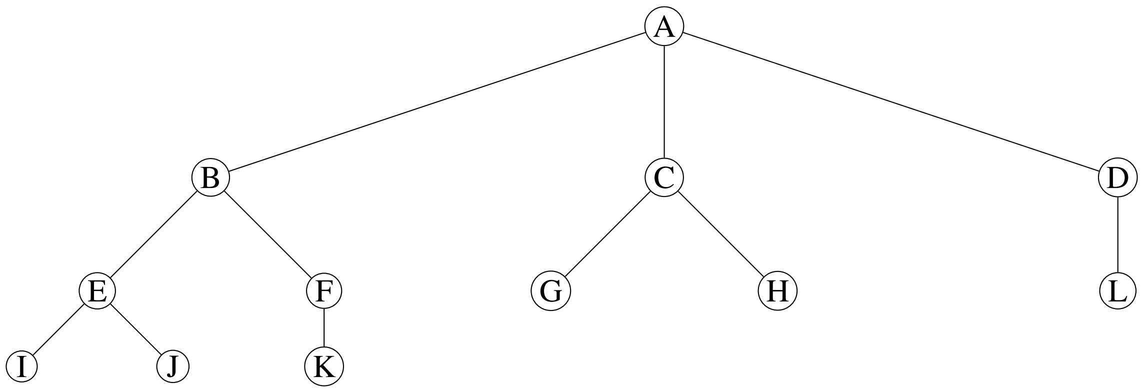

Figure 1.1: A branching metric space

Example 2. Examples of geodesic spaces that are branching are Banach spaces with norms that are not strictly convex, for example, $L^\infty(M)$ where M is a closed Riemannian manifold (Section 2.1).

Example 3. Another famous example of branching metric space is the *Hawaiian earrings*. It consists of a family of $\mathbb{S}^1_i := \mathbb{S}^1\left(\frac{1}{i}\right)$, circles of radius $\frac{1}{i}$ glued at a certain point p (see Figure 1.2). It is a geodesic metric space but geodesics that pass through p branch.

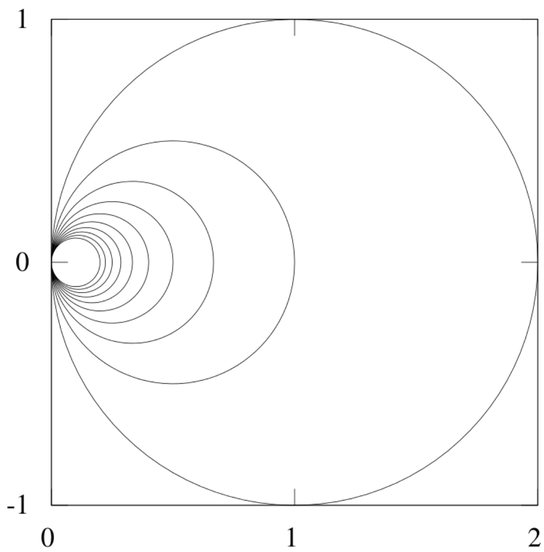

Figure 1.2: The Hawaiian Earrings

To finish this section, we briefly want to introduce the notion of *dimension* of a metric space because it will be used in the thesis. If $\mathcal{U}$ is an open cover of X, its *order* is defined as the minimum value n such that any $p \in X$ is contained in, at most, n members of $\mathcal{U}$.

Definition 3 (Topological dimension of a metric space)**.** We define the *topological dimension* of X, $\dim X$ or $\dim_T X$, as the minimum value n such that any open cover $\mathcal{U}$ of X has an open refinement with order less or equal than $n+1$.

For example, the topological dimension of a n–Riemannian manifold coincides with n.

Remark. There are other dimensions known for metric spaces, such as the Hausdorff dimension. The relation between the topological and Hausdorff dimension is the following inequality:

$$\dim_T X \leq \dim_{\mathcal{H}} X.$$

1.2 Curvature on geodesic metric spaces

Metric spaces can be endowed with many geometric restrictions, acquiring a lot of regularity as a consequence. One of them is *curvature* in term of bounds. As is it said in [19, Introduction 4.1.1]: *"Loosely speaking, curvature bounds guarantee a certain degree of convexity of concavity for distance functions"*. In a deeper sense, we will show some convexity considerations for distance functions in Section 1.3.

We are going to define the curvature by three definitions that are equivalent [19, Section 4.3] . Moreover, we are going to follow [19] for the definitions and structure. It is important to remark that although we are going to define “a space having nonpositive (resp. nonnegative) curvature”, we will not define a notion of curvature alone, and neither a numerical value to it.

1.2.1 Comparisons for distance functions

Let (X, dist_X) be a geodesic space and fix $p \in X$. We can define the real valued function

$$\mathrm{dist}_p(\cdot) := \mathrm{dist}_X(p, \cdot).$$

Furthermore, letting $\gamma\colon [0,T] \to X$ be a minimizing geodesic such that $\gamma(0) = a$ and $\gamma(T) = b$, then we can introduce the following function

$$g(t) := \mathrm{dist}_p \circ\gamma(t) = \mathrm{dist}_p(\gamma(t)). \tag{1.1}$$

We will call such functions 1*–dimensional distance functions*.

In this section we are always going to use the Euclidean space as the “reference” space due to its flatness. The curvature notion will be established in terms of how much convex or concave $g(t)$ is in terms of a suitable 1–dimensional Euclidean distance function.

For that purpose, we choose a segment of the same length as $T = |ab| = |\gamma(0)\gamma(T)|$ but in the Euclidean space and a point $\widetilde{p}$ ”positioned in the same way as p is positioned with respect to γ”. Formally speaking, we choose an Euclidean comparison segment from $\widetilde{a}$ to $\widetilde{b}$ of length $T = |\widetilde{a}\widetilde{b}|$ and $\widetilde{p}$ such that

$$|\widetilde{a}\widetilde{p}| = \mathrm{dist}_p(a)$$
$$|\widetilde{b}\widetilde{p}| = \mathrm{dist}_p(b).$$

For clarity, we will parametrize this segment as $\widetilde{\gamma}(t)$ such that $\widetilde{\gamma}(0) = \widetilde{a}$ and $\widetilde{\gamma}(T) = \widetilde{b}$.

Remark. As Burago, Burago and Ivanov pointed out in [19], this comparison configuration is unique up to a rigid motion.

Definition 4 ([19], Definition 4.1.1)**.** The *comparison function* for $g(t)$ is

$$\widetilde{g}(t) = |\widetilde{p}\widetilde{\gamma}(t)|, \tag{1.2}$$

the Euclidean distance from $\widetilde{p}$ restricted to a comparison segment from $\widetilde{a}$ to $\widetilde{b}$.

The convention will be that distance functions for *nonpositively* (resp. *nonnegatively*) curved spaces must be "more" convex (resp. "more" concave) than for the Euclidean plane. We are going to establish that conditions by the inequality $\widetilde{g}(t) \leq g(t)$ (resp. $\widetilde{g}(t) \geq g(t)$).

Definition 5 ([19], Definition 4.1.2)**.** We say that (X, dist_X) is *nonpositively* (resp. *nonnegatively*) *curved* if every point in X has a neighbourhood such that, whenever a point $p \in X$ and a geodesic γ lies within this neighbourhood, the comparison function $\widetilde{\gamma}(t)$ for the 1–dimensional distance function $g(t) = \mathrm{dist}_p(\gamma(t))$ satisfies $\widetilde{g}(t) \leq g(t)$ (resp. $\widetilde{g}(t) \geq g(t)$) for all $t \in [0, T]$.

Example 4 ([19], Example 4.1.3)**.** Glue together three copies of the ray $[0, \infty) \subset \mathbb{R}$ at the point 0. The resulting space $R_{(3)}$ has nonpositive curvature.

There is a proof of this statement in [19, Proof of Example 4.1.3].

Example 5 ([19], Example 4.1.5)**.** Consider $\mathbb{R}^2$ with the norm $||u|| = |x| + |y|$ where x, y are the Cartesian coordinates of v. $(\mathbb{R}^2, || \cdot ||)$ is neither nonnegatively nor nonpositively curved.

There is a proof of this statement in [19, Proof of Example 4.1.5].

1.2.2 Distance comparison for triangles

A more visual way to understand the curvature notion of a metric space is the one given by *comparison of triangles*. A *triangle* in X is a collection of three points a, b and $c \in X$ (so called *vertices*) connected by three minimizing geodesics (called *sides*). For clarity we will call this minimizing geodesic with the same notation as [19]: $[ab]$, $[bc]$ and $[ca]$ as well as their length $|ab|$, $|bc|$ and $|ca|$. By $\angle abc$ we denote the angle between the shortest paths $[ba]$ and $[bc]$ at b (if this angle is well-defined).

We need to extend the Euclidean notion of *angle* to the metric context.

Definition 6 ([19], Definition 3.6.25 & 3.6.26)**.** Let x, y, z be three distinct points in a metric space (X, dist_X). The *comparison angle* xyz, denoted by $\widetilde{\angle} xyz$ or $\widetilde{\angle}(x, y, z)$ is defined by

$$\widetilde{\angle} xyz = \arccos \frac{\mathrm{dist}_X(x,y)^2 + \mathrm{dist}_X(y,z)^2 - \mathrm{dist}_X(x,z)^2}{2\,\mathrm{dist}_X(x,y)\,\mathrm{dist}_X(y,z)}$$

Let $\alpha\colon [0, \epsilon) \to X$ and $\beta\colon [0, \epsilon) \to X$ be two paths in a length space X emanating from the same point $p = \alpha(0) = \beta(0)$. We define the angle $\angle(\alpha, \beta)$ between α and β as

$$\angle(\alpha, \beta) = \lim_{s,t \to 0} \widetilde{\angle}(\alpha(s), p, \beta(t))$$

if the limit exists.

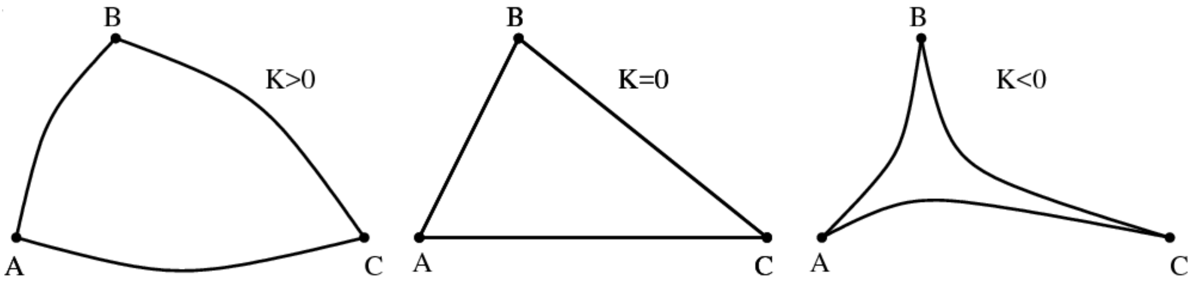

Figure 1.3: [19, Chapter 4, page 108] Different triangles in terms of the curvature.

In our context, $\angle abc$ is the one formed by the geodesics $\alpha = [ba]$, $\beta = [bc]$ and $p = b$.

Once we have a triangle $\triangle abc \subset X$, we construct a *comparison* triangle $\triangle \tilde{a}\tilde{b}\tilde{c}$ in the Euclidean plane with the same lengths of sides,

$$|ab| = |\widetilde{a}\widetilde{b}|, \qquad |bc| = |\widetilde{b}\widetilde{c}|, \qquad |ac| = |\widetilde{a}\widetilde{c}|.$$

Definition 7 ([19], Definition 4.1.9)**.** The space (X, dist_X) is a space of *nonpositive* (resp. *nonnegative*) curvature if in some neighbourhood of each point the following holds:

For every $\triangle abc$ and every point $d \in [ac]$, one has $|db| \leq |\widetilde{d}\widetilde{b}|$ (resp. $|db| \geq |\widetilde{d}\widetilde{b}|$) where $\widetilde{d}$ is the point on the side $[\widetilde{a}\widetilde{c}]$ of a comparison triangle $\triangle \widetilde{a}\widetilde{b}\widetilde{c}$ such that $|ad| = |\widetilde{a}\widetilde{d}|$.

Remark. The assertion of this previous definition regarding nonpositive curvature is known as *Triangle condition* or CAT (0) *condition*. For that reason, spaces fitting this definition are called CAT (0)-spaces.

CAT stands for the comparison of Cartan–Alexandrov–Toponogov and (0) means that we impose zero as the upper curvature bound (i.e., we are using the Euclidean plane). As the reader may imagine, we can replace the Euclidean plane with other spaces with constant curvature (such as spheres), so we can generalize the definition to CAT (κ)-spaces, $\kappa \in \mathbb{R}$. See Figure 1.3. For more information, we encourage the reader to check [3, 19, 21, 38].

Example 6 (Examples of CAT (κ)-spaces)**.**

- The n–Euclidean space $\mathbb{E}^n$ with the canonical metric is a CAT(0)-space.
- The round unit n–sphere $\mathbb{S}^n$ is a CAT (1)-space.
- The n–Hyperbolic space $\mathbb{H}^n$ with the canonical metric is a CAT (-1)-space.
- We denote by a *metric segment* (of length ℓ) a metric space isometric to a segment $[0, \ell]$. We define a *metric graph* as the result of gluing a disjoint collection of metric segments $\{E_i\}$ and points $\{v_j\}$ along an equivalence relation $\mathcal{R}$ defined on the union of the set $\{v_j\}$ and the set of the endpoints of the segments.

  A geodesic metric space (X, dist_X) is a *tree* if every triangle is a *tripoid*, i.e., for every three points a, b, $c \in X$ there exists $d \in X$ such that the geodesic segments $[ca]$ and $[cb]$ intersect in the segment $[cd]$ and also $d \in [ab]$. See Figure 1.4.

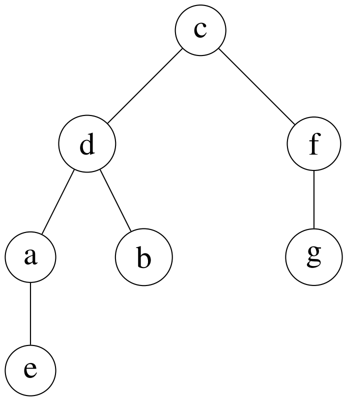

Figure 1.4: A metric tree

Proposition 1.1 ([21]). *A metric graph* (X, dist_X) *is a CAT (0) space if and only if it is a tree.*

The other inequality gives us another family of interesting spaces. *Alexandrov spaces* are general length spaces with a lower curvature bound. They have been widely studied due to their properties. For the interested reader, here are some important references [2, 20, 34, 36, 75].

Here we present some examples of Alexandrov spaces:

- Euclidean spaces.
- A convex set in an Alexandrov space is obviously an Alexandrov space.
- Riemannian manifolds with sectional curvature bounded from below.

1.2.3 Angle comparison for triangles

The final equivalent definition of the curvature on metric spaces seems to be a reformulation of the triangle condition, but, surprisingly the proving their equivalence require some work.

If we observe Figure 1.3, it would be obvious to think that ”fat” triangles should have large angles and “skinny” triangles should have smaller ones. The following definition is the formalization of that intuition:

Definition 8 ([19], Definition 4.1.15). (X, dist_X) is a space of *nonpositive curvature* if every point of X has a neighbourhood such that, for every triangle $\triangle abc$ contained in this neighbourhood, the angles $\angle bac$, $\angle cba$ and $\angle abc$ are well defined and satisfy the following inequalities:

$$\angle bac \leq \angle \tilde{b}\tilde{a}\tilde{c}, \quad \angle abc \leq \angle \tilde{a}\tilde{b}\tilde{c}, \quad \angle bca \leq \angle \tilde{b}\tilde{c}\tilde{a}. \tag{1.3}$$

Conversely, (X, dist_X) is a space of *nonnegative curvature* if every point of X as a neighbourhood such that, for every triangle $\triangle abc$, the above angles are also well defined and we have the opposite inequalities in (1.3). In addition, the following holds: for any two shortest path $[pq]$ and $[rs]$ where r is an inner point of $[pq]$, one has

$$\angle prs + \angle srq = \pi.$$

1.3 Some convexity considerations about the distance function of a metric space

Three assumptions are imposed on a function $f : X \times X \to \mathbb{R}$ to be a distance and turn (X, f) into a metric space: symmetry, positiveness and the triangle inequality. Apart from those, any other constraint on f gives extra structure to the space X, as we have presented in Section 1.2 with the notion of curvature of metric spaces. Martin Kell did it in [57], studying some convexity properties of the distance function and using them on the initial hypothesis of the majority of the theorems of that paper; those convexity properties are essential for some of the results of Section 3.2. We present here most of these definitions that also appear in [57, Section 1].

We begin recalling a basic metric definition:

Definition 9 (Midpoints)**.** We say that (X, dist) admits *midpoints* if , for every $x, y \in X$, there is $m(x, y) \in X$ such that

$$\mathrm{dist}(x, m(x, y)) = \mathrm{dist}(y, m(x, y)) = \frac{1}{2}\,\mathrm{dist}(x, y).$$

Midpoints are the key ingredient for the following three definitions:

Definition 10 (p–convex, p–Busemann curvature and uniformly p–convex)**.** Let (X, dist) be a metric space that admits midpoints.

1. X is p*–convex* for some $p \in [1, \infty]$ if, for each triple $x, y, z \in X$ and each midpoint $m(x, y)$ of x and y,

   $$\mathrm{dist}(m(x, y), z) \le \left(\frac{1}{2}\,\mathrm{dist}(x, z)^p + \frac{1}{2}\,\mathrm{dist}(y, z)^p\right)^{1/p}.$$

   The space X is called *strictly* p*–convex* for $p \in (1, \infty]$ if the inequality is strict for $x \neq y$ and *strictly* 1*–convex* if the inequality is strict whenever $\mathrm{dist}(x, y) > |\,\mathrm{dist}(x, z) - \mathrm{dist}(y, z)|$.

2. X satisfies the p*–Busemann curvature condition* if, for all $x_0, x_1, y_0, y_1 \in X$ with midpoints $m_x = m(x_0, x_1)$ and $m_y = m(y_0, y_1)$,

   $$\mathrm{dist}(m_x, m_y) \le \left(\frac{1}{2}\,\mathrm{dist}(x_0, y_0)^p + \frac{1}{2}\,\mathrm{dist}(x_1, y_1)^p\right)^{1/p}$$

   for some $p \in [1, \infty]$. If X satisfies the p–Busemann condition, we say that (X, dist) is p*–Busemann*. In particular, if $p = 1$, we say that (X, dist) is *Busemann*.

   It turns out that (X, dist) is a Busemann space if and only if

   $$\mathrm{dist}(m(x, z), m(x, y)) \le \frac{1}{2}\,\mathrm{dist}(z, y).$$

3. X is *uniformly p–convex* for some $p \in [1, \infty]$ if, for all $\epsilon > 0$, there exists $\rho_p(\epsilon) \in (0, 1)$ such that, for every $x, y, z \in X$ satisfying

$$\operatorname{dist}(x, y) > \epsilon \left(\frac{1}{2} \operatorname{dist}(x, z)^p + \frac{1}{2} \operatorname{dist}(y, z)^p \right)^{1/p}, \text{ for some } p > 1,$$

or

$$\operatorname{dist}(x, y) > |\operatorname{dist}(x, z) - \operatorname{dist}(y, z)| + \epsilon \left(\frac{1}{2} \operatorname{dist}(x, z) + \frac{1}{2} \operatorname{dist}(y, z) \right), \text{ for } p = 1,$$

the following inequality holds:

$$\operatorname{dist}(m(x, y), z) \leq (1 - \rho_p(\epsilon)) \left(\frac{1}{2} \operatorname{dist}(x, z)^p + \operatorname{dist}(y, z)^p \right)^{1/p}.$$

For example, every $\mathrm{CAT}(0)$–space is uniformly 2–convex.

Remark. By [57, Lemma 1.4., Corollary 1.5.], the following assertions hold taking into account the previous definitions:

- A uniformly p–convex metric space is uniformly p'–convex for all $p' \geq p$.
- Assume (X, dist) is Busemann. Then (X, dist) is strictly (resp. uniformly) p–convex for some $p \in [1, \infty]$ if and only if it is strictly (resp. uniformly) p–convex for all $p \in [1, \infty]$.
- Any $\mathrm{CAT}(0)$–space is both Busemann and uniformly 2–convex, thus uniformly p–convex for every $p \in [1, \infty]$.

Finally, another interesting property we can impose to the metric spaces is the *reflexivity*, an intersection phenomena with multiple reformulations. We have decided to display the following due to its readability:

Definition 11 (Reflexive metric space, Definition 2.1. [57])**.** Let I be a directed set. A metric space (X, dist) is *reflexive* if, for every non–increasing family $\{C_i\}_{i \in I} \subset X$ of non–empty bounded closed convex subsets (i.e. $C_i \subset C_j$ whenever $i \geq j$), we have

$$\bigcap_{i \in I} C_i \neq \emptyset.$$

1.4 Submersions and submetries

Maps between spaces are always a fruitful configuration in order to extract properties for those spaces. In the case of Riemannian manifolds, there are two big families of functions: embeddings/immersions and submersions. Here, we present the definition and some examples of *Riemannian submersions* and its generalization to metric spaces, the so-called *submetries*.

Definition 12 (Riemannian submersion)**.** A differentiable map $\pi\colon M^{m+n} \to B^n$ is called a *submersion* if π is surjective, and for all $p \in M$, $d\pi_p\colon T_pM \to T_{\pi(p)}B$ has rank n. If M and B have Riemannian metrics, the submersion π is said to be *Riemannian* if, for all $p \in M$, $d\pi_p\colon T_pM \to T_{\pi(p)}B$ preserves the lengths of vectors orthogonal to F_p, where $F_p := \pi^{-1}(p)$.

Example 7. Let

$$\pi\colon \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$$
$$(p_1, p_2) \mapsto p_1$$

be the projection onto the first factor. Thus, π is a Riemannian submersion.

Example 8. Let (M, g) be a Riemannian manifold and G a Lie group that acts isometrically, freely and properly on M. Let $N = M/G$ be the quotient space equipped with the quotient metric. Then, the projection $\pi\colon M \to N$ is a Riemannian submersion.

The definition of a Riemannian submersion was generalized to metric spaces under the notion of the *submetry*. It is a *purely metric version* of Riemannian submersions. The reader can find more information in [9, 10, 42]. Here, we also present the definition:

Definition 13 (Submetry)**.** A *submetry* between metric spaces is a map $\pi\colon X \to B$ such that for every $p \in X$, any closed ball $B(p, r)$ of radius $r > 0$ centered at p maps onto the ball $B(\pi(p), r)$.

Indeed, the generalization was proved on [10]:

Theorem 1.1 ([10], Theorem A)**.** *Let $\Phi\colon M \to B$ a submetry between Riemannian manifolds. Then Φ is a $C^{1,1}$ Riemannian submersion.*

Moreover, the existence of submetries imposes certain curvature restrictions:

Theorem 1.2 ([10], Theorem C)**.** *Let $\pi\colon M \to N$ be a submetry of complete Riemannian manifolds where M has nonnegative sectional curvature. Then*

1. *N has also nonnegative curvature.*
2. *If N is compact and M is flat, then N is flat.*

For the interested reader, we recommend [42] for the relation between submetries and Alexandrov spaces.

1.5 Optimal Transport and Wasserstein type spaces

The optimal transport problem first appeared in 1781 in the work of Gaspard Monge [70]. The setting is quite simple: we search for the optimal way to transport one pile of sand to another. In other words, imagine that the pile of sand is modeled by a probability measure μ and the new location is ν, another probability measure. So we have to clarify how to move the mass of μ. This problem involves a cost c of moving a point x (whatever it means in terms of sand) to its new location $T(x)$. Then, the problem consists in finding certain *plan* in which to move all the mass from μ to ν minimizing the cost c.

More formally: let (X, dist_X) be a complete and separable metric space and let μ, ν be two probability measures supported on X. Monge's optimal transport problem consists on minimizing:

$$\int c(x, T(x))\, d\mu(x), \tag{1.4}$$

among all measurable maps $T\colon X \to X$ such that $T_{\#}\mu = \nu$.

Remark. In the case of Monge's problem, the cost function c is settled to be $\operatorname{dist}_X(x, T(x))^2$.

Kantorovich presented a new formulation of the optimal transport problem, in some sense, weaker than the Monge's one: instead of imposing the existence of some function T that sends one measure μ to another ν, he allows the possibility to split mass, i.e., he proposed to minimize the following functional:

$$\pi \mapsto \int \operatorname{dist}_X(x, y)^2 d\pi, \tag{1.5}$$

among all *admissible measures* $\pi \in \Gamma(\mu, \nu)$. More, formally:

Definition 14 (Transference plan)**.** A *transference plan*, *admissible plan* or *admissible measure* between two positive measures $\mu, \nu \in \mathcal{P}(X)$ is a finite positive measure $\pi \in \mathcal{P}(X \times X)$ (the set of probability measures on $X \times X$) which satisfies that, for all Borel subsets A, B of X,

$$\pi(A \times X) = \mu(A), \quad \text{and} \quad \pi(X \times B) = \nu(B).$$

Remark. The functional (1.5) is linear and the set $\Gamma(\mu, \nu)$ of admissible measures is convex and closed in the weak topology [4]. A measure that minimizes the functional will be called *optimal* and the set of all of them will be denoted as $\operatorname{Opt}(\mu, \nu)$.

1.5.1 Wasserstein space

Let (X, dist) be a geodesic space and $\mathcal{P}(X)$ denote the set of probability measures on X and $\mathcal{P}_p(X)$ the ones with finite p–moment.

Note that we require $1 = |\mu| = |\nu| = \pi(X \times X)$, so we are not considering all measures of the product space. We denote by $\Gamma(\mu, \nu)$ the set of transference plans between the measures μ and ν. Then, we define the p–Wasserstein distance for $p \geq 1$ between two probability measures as

$$W_p(\mu, \nu) := \left(\min_{\pi \in \Gamma(\mu,\nu)} \int_{X \times X} \operatorname{dist}(x, y)^p d\pi(x, y) \right)^{\frac{1}{p}}.$$

The metric space $(\mathcal{P}_p(X), W_p)$ is denoted as the *p–Wasserstein space of X*.

Now, in order to give a general perspective about these spaces, we will present some theorems regarding the topological structure of the Wasserstein space:

Theorem 1.3 ([4], Theorem 3.7)**.** *Let (X, dist) be a complete and separable metric space. Then $(\mathcal{P}_2(X), W_2)$ is complete and separable as well. In addition, any measure may be approximated by a sequence of totally atomic probability measures (measures with support formed by points). Also, the following are equivalent:*

1. *A sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_2(X)$, converges (via some subsequence) in the W_2 distance to a measure μ.*

2. *$\mu_n \rightharpoonup \mu$ and $\int \operatorname{dist}(\cdot, x_0)^2 d\mu_n \to \int \operatorname{dist}(\cdot, x_0)^2 d\mu$ for some $x_0 \in X$, where we define $\mu_n \rightharpoonup \mu$ as narrowly convergence to a probability measure μ, if:*

$$\int \phi \, d\mu_n \to \int \phi \, d\mu \text{ as } n \to \infty, \forall \phi \in C_b(X),$$

where $C_b(X)$ is the space of bounded continuous functions in X.

Theorem 1.4 ([91], Remark 6.19; [4], Theorem 3.10 & Proposition 3.16)**.** *The space* (X, dist) *is compact if and only if* $(\mathcal{P}_2(X), W_2)$ *is compact. If* (X, dist) *is only locally compact, then* $(\mathcal{P}_2(X), W_2)$ *is not locally compact.*

If (X, dist) *is a geodesic space, then* $(\mathcal{P}_2(X), W_2)$ *is geodesic as well. Also, the following are equivalent:*

1. *The map* $t \mapsto \mu_t \in \mathcal{P}_2(X)$ *is a geodesic.*
2. *There exists some measure* $\mu \in \mathcal{P}_2(\mathrm{Geo}(X))$ *such that* $(e_0, e_1)_\# \in \mathrm{Opt}(\mu_0, \mu_1)$ *and*

$$\mu_t = e_{t\#}\mu,$$

where e_t *is the evaluation map defined as*

$$\begin{aligned} e_t \colon\ \mathrm{Geo}(X) &\to X \\ \gamma &\mapsto e_t(\gamma) = \gamma(t). \end{aligned}$$

If (X, dist) *is a non-branching geodesic space, then* $(\mathcal{P}_2(X), W_2)$ *is also non-branching. Moreover, if* $t \mapsto \mu_t \in \mathcal{P}_2(X)$ *is a geodesic, then for all* $t \in (0,1)$ *there exists a unique optimal plan in* $\mathrm{Opt}(\mu_t, \mu_1)$ *and it is induced by a map from* μ_t*.*

Generalized and Signed Wasserstein distance

A natural question regarding the generalization of the Wasserstein distance and the Wasserstein space involves measures with different masses or, even more, signed measures. In that sense, Mainini and Piccoli developed on [67, 78] the notion of Generalized Wassertein distance and later on they added the signed restriction.

We used both distances on Chapter 4, so we present here the definitions:

Definition 15. [Generalized Wasserstein distance, [67, 78]] Let μ, ν be two positive measures in $\mathcal{M}(X)$ with possibly different mass. The generalized Wasserstein distance between μ and ν is given for $p \geq 1$ $a > 0$ and $b > 0$ by

$$W_p^{a,b}(\mu, \nu) = \left(\inf_{\substack{\tilde{\mu}, \tilde{\nu} \in \mathcal{M}(X) \\ |\tilde{\mu}| = |\tilde{\nu}|}} a^p (|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|)^p + b^p W_p^p(\tilde{\mu}, \tilde{\nu}) \right)^{1/p}.$$

Definition 16. [Signed Generalized Wasserstein distance, [67, 78]] Let $\mu, \nu \in \mathcal{M}^s(X)$. We define their distance by

$$\mathbf{W}_1^{a,b}(\mu, \nu) := W_1^{a,b}(\mu_+ + \nu_-, \mu_- + \nu_+),$$

where $\mathcal{M}^s(X)$ are the signed and finite measures on X.

With this definition in mind, we can introduce the following norm:

Definition 17. For $\mathcal{M}^s(X)$ and $a > 0$, $b > 0$, we define

$$||\mu||^{a,b} := \mathbf{W}_1^{a,b}(\mu, 0) = W_1^{a,b}(\mu_+, \mu_-),$$

where μ_+ and μ_- are any positive measures of $\mathcal{M}(X)$ such that $\mu = \mu_+ - \mu_-$.

This norm turns $(\mathcal{M}^s(X), ||\cdot||^{a,b})$ into a normed vector space [78, Proposition 21]. Moreover, for any $a, b > 0$ the norm $||\cdot||^{a,b}$ is equivalent $||\cdot||^{1,1}$ [78, Proposition 23]. We used that result on Chapter 4 in order to facilitate the computations.

1.5.2 Orlicz–Wasserstein space

In this section, we present the *Orlicz–Wasserstein space*, a generalization of the Wasserstein space. Let $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing, continuous function. Assume ϑ admits a representation $\vartheta = \varphi \circ \psi$ as a composition of a convex and a concave function φ and ψ, respectively. This includes all $\mathcal{C}^2$ functions [82, Example 1.3.].

Definition 18 (L^ϑ–Wasserstein space and distance)**.** Let (X, dist) be a complete separable metric space. The L^ϑ–Wasserstein space $\mathcal{P}_\vartheta(X)$ is defined by all probability measures μ in X such that

$$\int_X \varphi\left(\frac{1}{t}\psi(\mathrm{dist}(x,y))\right) d\mu(x) < \infty.$$

The L^ϑ–Wasserstein distance of two probability measures $\mu, \nu \in \mathcal{P}_\vartheta(X)$ is defined as

$$W_\vartheta(\mu,\nu) = \inf\left\{ t > 0 : \inf_{\pi\in\Gamma(\mu,\nu)} \int_{X\times X} \varphi\left(\frac{1}{t}\psi(\mathrm{dist}(x,y))\right) d\pi(x,y) \leq 1 \right\}.$$

The function W_ϑ is a complete metric on $\mathcal{P}_\vartheta(X)$ (see [82], Proposition 3.2). The metric space $(\mathcal{P}_\vartheta(X), W_\vartheta)$ is known as the ϑ*-Orlicz–Wasserstein space of* X.

Notice that for every $x \in X$, the probability measure δ_x belongs to $\mathcal{P}_\vartheta(X)$. Therefore, we can embed the metric space X inside its Orlicz–Wasserstein space by mapping $x \mapsto \delta_x$. In addition, this map is an isometric embedding if and only if $\psi \equiv \mathrm{Id}$ and $\varphi(1) = 1$. Moreso, if $\varphi = \mathrm{dist}^p$, then we obtain the usual p–Wasserstein space.

On this spaces, we also have some of the properties we extract from the usual Wasserstein space. Here, we present a similar result to Theorem 1.4:

Proposition 1.2 ([56], Proposition A.6)**.** *Let* (X, dist) *be a geodesic space, then* $(\mathcal{P}_\vartheta(X), W_\vartheta)$ *is also geodesic.*

We recommend [56, 82] in order to obtain more information about these spaces.

1.5.3 Space of Persistence Diagrams

The importance of Data Science has been reaching every field in mathematics. Topology and geometry are facing this wave of information analysis with new perspectives that are helping to face some challenges concerning high dimensionality of the data sets or its inner structure. One of this ongoing fields is Topological Data Analysis (TDA).

The key tool used on TDA is the *persistence diagram*. This set of point capture all the output of the data analysis and has very useful graphical representations (see Fig 1.5) for understanding the set. In addition, we can endow the space of persistence diagrams with Wasserstein type distances, that in the daily work with this objects help with the computations due to their useful implementations in different coding languages.

We begin this section presenting the definition of a persistence diagram:

Definition 19 (Persistence diagram, [18])**.** A *persistence diagram* is a function from a countable set I to $\mathbb{R}^2_<$, i.e. $D\colon I \to \mathbb{R}^2_<$, where $\mathbb{R}^2_< = \{(x,y) \in \mathbb{R}^2 : x < y\}$.

In this definition, all the points have multiplicity one. Other authors suggest considering persistence diagrams as multisets of points, i.e. sets of points where we can repeat points (see [27, 26, 69, 89]). This consideration is closer to the performance of the persistence diagrams in the TDA setting as various homological features can have the same birth and death.

Moreover, Che, Galaz-García, Guijarro and Membrillo-Solis defined in [27, 26] the notion of generalized persistence diagrams $\mathrm{Dgm}(X, A)$ (in this setting the persistence diagram defined in Definition 19 will be $\mathrm{Dgm}_2(\mathbb{R}^2, \Delta)$ where $\Delta \subset \mathbb{R}^2$ denotes the diagonal) extending the notion of persistence diagrams beyond the Euclidean setting and presenting a general definition for points in metric spaces.

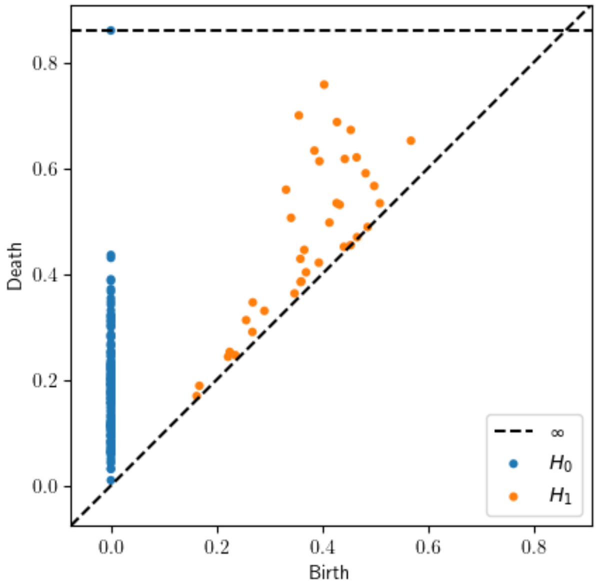

Figure 1.5: A Persistence Diagram of a point cloud over a 2–dimensional sphere

We construct a Wassertein type distance on the space of persistence diagrams:

Definition 20 (Partial matching, [18]). Let $D_1 \colon I_1 \to \mathbb{R}^2_<$ and $D_2 : I_2 \to \mathbb{R}^2_<$ be persistence diagrams. A *partial matching* between them is a triple (I_1', I_2', f) such that $I_1' \subseteq I_1$, $I_2' \subseteq I_2$, and $f \colon I_1' \to I_2'$ is a bijection.

Definition 21 (Cost of a partial matching, [18]). Let $D_1 \colon I_1 \to \mathbb{R}^2_<$ and $D_2 \colon I_2 \to \mathbb{R}^2_<$ be persistence diagrams and (I_1', I_2', f) a partial matching between them. We endow $\mathbb{R}^2$ with the infinity metric $\mathrm{dist}_\infty(a, b) = ||a - b||_\infty = \max(|a_x - b_x|, |a_y - b_y|)$. Observe that, for $a \in \mathbb{R}^2_<$, we have that $\mathrm{dist}_\infty(a, \Delta) = \inf_{t \in \Delta} \mathrm{dist}_\infty(a, t) = (a_y - a_x)/2$. We denote by $\mathrm{cost}_p(f)$ the *p–cost of f*, defined as follows. For $p < \infty$, let

$$\mathrm{cost}_p(f) = \Big(\sum_{i \in I_1'} \mathrm{dist}_\infty(D_1(i), D_2(f(i)))^p + \sum_{i \in I_1 \setminus I_1'} \mathrm{dist}_\infty(D_1(i), \Delta)^p + \sum_{I_2 \setminus I_2'} \mathrm{dist}_\infty(D_2(i), \Delta)^p \Big)^{1/p},$$

and for $p = \infty$, let

$$\mathrm{cost}_\infty(f) = \max\Big\{ \sup_{i\in I_1'} \mathrm{dist}_\infty(D_1(i), D_2(f(i))), \sup_{i\in I_1\setminus I_1'} \mathrm{dist}_\infty(D_1(i), \Delta), \sup_{i\in I_2\setminus I_2'} \mathrm{dist}_\infty(D_2(i), \Delta)\Big\}.$$

If any of the terms in either expression is unbounded, we declare the cost to be infinity.

Now we can define the distance functions and the metric space of persistence diagrams:

Definition 22 (p–Wasserstein distance and bottleneck distance of persistence diagrams, [30]). Let $1 \le p \le \infty$ and D_1, D_2 persistence diagrams. Define

$$\tilde{w}_p(D_1, D_2) = \inf\{\mathrm{cost}_p(f) \colon f \text{ is a partial matching between } D_1 \text{ and } D_2\}.$$

Let (Dgm_p, w_p) denote the metric space of persistence diagrams D such that $\tilde{w}_p(D, \emptyset) < \infty$ with the relation $D_1 \sim D_2$ if $\tilde{w}_p(D_1, D_2) = 0$, where $\emptyset$ is the unique persistence diagram with empty indexing set. The metric w_p is called the *p–Wasserstein distance* and w_∞ is called the *bottleneck distance*.

Recall that in Section 1.2.2 we present the notion of an Alexandrov space. If we endow the space of persistence diagrams (Dgm_2, w_2) with the 2–Wasserstein metric, then

Theorem 1.5 (Turner, Mileyko, Mukherjee & Harer, [89]). *The space of Persistence Diagrams (Dgm_2, w_2) is a non-negatively curved Alexandrov space.*

For generalized persistence diagrams, Che, Galaz-García, Guijarro and Membrillo-Solís, proved a similar result:

Theorem 1.6 (Che, Galaz-García, Guijarro & Membrillo-Solís, [26], Theorem B). *If X is a proper Alexandrov space with non-negative curvature, then $\mathrm{Dgm}_2(X, A)$ is an Alexandrov space with non-negative curvature.*

1.6 Reach of a subset

In [32], Federer studied properties about convex subsets of the n–Euclidean space. In that sense, he defined the concept of the *reach of a subset* A as some kind of measurement of the ϵ–neighbourhood around A only containing points of the ambient space with a unique metric projection on A.

Definition 23 (Unique points set and reach, [32]). Let (X, dist) be a metric space and $A \subset X$ a subset. We define the set of points having a unique metric projection in A as

$$\mathrm{Unp}(A) = \{x \in X : \text{there exists a unique } a \text{ such that } \mathrm{dist}(x, A) = \mathrm{dist}(x, a)\}.$$

For $a \in A$, we define the *reach* of A at a, denoted by $\mathrm{reach}(a, A)$, as

$$\mathrm{reach}(a, A) = \sup\{r \ge 0 : B_r(a) \subset \mathrm{Unp}(A)\}.$$

Finally, we define the *global reach* by

$$\mathrm{reach}(A) = \inf_{a\in A} \mathrm{reach}(a, A).$$

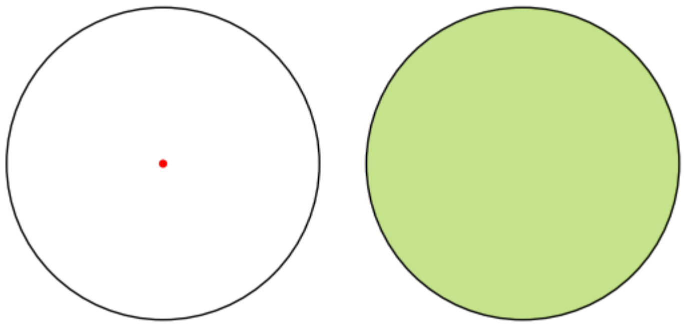

Figure 1.6: $\mathrm{reach}(\mathbb{S}^1 \subset \mathbb{R}^2) = 1$ due to its center while $\mathrm{reach}(\mathbb{D}^1 \subset \mathbb{R}^2) = \infty$.

The intuitive idea is that $\mathrm{reach}(A) = 0$ if and only if we do *not* have an ϵ–neighbourhood of A admitting a metric projection onto A. Conversely, $\mathrm{reach}(A) = \infty$ will occur if and only if the entirety of X admits a metric projection into A.

Remark. Let $M \subset \mathbb{E}^n$ be a compact C^2–submanifold of the n–Euclidean space. Due to the normal neighbourhood theorem, we obtain that

$$\mathrm{reach}(M \subset \mathbb{E}^n) > 0.$$

Example 9. Let $\mathbb{S}^1 \subset \mathbb{R}^2$. As there exists a point $0 \in \mathbb{R}^2$, that is the center, at the same distance of every point of $\mathbb{S}^1$, we have that

$$\mathrm{reach}(\mathbb{S}^1 \subset \mathbb{R}^2) = 1,$$

whereas in the case of $\mathbb{D}^2 \subset \mathbb{R}^2$, we obtain that

$$\mathrm{reach}(\mathbb{D}^2 \subset \mathbb{R}^2) = \infty.$$

With these two subsets of $\mathbb{R}^2$, we can construct a sequence of subsets to prove that reach is not continuous under Gromov–Hausdorff distance (see [19] for more information about that distance): let $\mathcal{C}_r = \{\mathbb{D}^2 \backslash \mathbb{D}^2_r\}$ such that $r \to 0$, where $\mathbb{D}^2$ is considered with radius 1 and $\mathbb{D}^2_r$ has radius r, then

$$\mathrm{reach}(\mathcal{C}_r \subset \mathbb{R}^2) = r, \text{ but } \mathrm{reach}(\mathbb{D}^2 \subset \mathbb{R}^2) = \infty.$$

The interested reader can look at [88], a survey by Christop Thälle with a rigorous review about the most relevant results in the area. More related work about positive reach and Riemannian geometry can be found in the papers written by Kapovitch and Lytchak [48, 65, 66].

Chapter 2

Filling Radius and the L^∞ space of a Riemannian manifold

We present some results concerning an invariantof Gromov's called *Filling Radius*. It is based on the tubular neighbourhoods of the *Kuratowski embedding*. We prove an upper and a lower bound for the Filling Radius as well as some estimations for the k–intermediate filling radius.

This chapter is based on the paper: Manuel Cuerno and Luis Guijarro. "Upper and lower bounds on the filling radius". In: *Indiana Univ. Math. J.* (2022). URL: https://arxiv.org/abs/2206.08032. Forthcoming.

2.1 The Kuratowski embedding

Let $(M^n, g, \operatorname{dist}_M)$ be a closed (compact without boundary) n–dimensional Riemannian manifold with a metric g and a distance function dist_M, and

$$L^\infty(M) = \{f\colon M \to \mathbb{R}\colon ||f||_\infty = \sup_{p\in M} |f(p)| < \infty\}.$$

We denote as $\operatorname{dist}_p(\cdot) = \operatorname{dist}_M(p,\cdot)\colon M \to \mathbb{R}$. There is a natural embedding, called *Kuratowski embedding*, of $M \hookrightarrow L^\infty(M)$ defined as:

$$\begin{aligned} \Phi\colon M &\to L^\infty(M) \\ p &\mapsto \operatorname{dist}_p . \end{aligned}$$

The Kuratowski embedding can also be extended to compact metric spaces.

Proposition 2.1. *The Kuratowski embedding is, indeed, an isometry, i.e.,*

$$\operatorname{dist}_\infty(\operatorname{dist}_p, \operatorname{dist}_q) = ||\operatorname{dist}_p - \operatorname{dist}_q||_\infty = \operatorname{dist}_M(p,q),\ \forall p,q \in M.$$

Proof. We split the proof in two inequalities:

1. $\geq$)

$$\begin{aligned} ||\operatorname{dist}_p - \operatorname{dist}_q||_\infty &= \sup_{x\in M} |\operatorname{dist}_p(x) - \operatorname{dist}_q(x)| \\ &\underset{x=q}{\geq} |\operatorname{dist}_p(q)| = \operatorname{dist}_M(p,q). \end{aligned}$$

2. $\leq$) Using the triangle inequality yields:

$$\begin{aligned}|| \operatorname{dist}_p - \operatorname{dist}_q ||_\infty &= \sup_{x\in M} |\operatorname{dist}_p(x) - \operatorname{dist}_q(x)| \\ &\leq \sup_{x\in M} |\operatorname{dist}_p(q)| = \operatorname{dist}_M(p, q).\end{aligned}$$

□

Remark. This isometry provides an interesting geometric property about $L^\infty(M)$: the geodesics of M are geodesics of $L^\infty(M)$. Even more, as $L^\infty(M)$ is a vector space, the segments are also geodesics. So, if we join p, $q \in M \subset L^\infty(M)$ by a segment, i.e. a minimizing geodesic in $L^\infty(M)$, we know the "size" of the segment in terms of the geodesic distance in M. Even more, the Kuratowski embedding is preserved under translations by an arbitrary function. So, indeed, we have infinite isometric copies of M into $L^\infty(M)$.

2.2 Filling radius

For a given coefficient group $\mathbb{F}$ (as Gromov stated in [39] $\mathbb{F} = \mathbb{Z}$ in the oriented case and $\mathbb{F} = \mathbb{Z}_2$ in the non-oriented), consider the homomorphism induced in n-homology by the inclusion

$$\iota_{r,*} : \mathrm{H}_n(M, \mathbb{F}) \to \mathrm{H}_n(U_r(M), \mathbb{F}),$$

where $U_r(M)$ denotes the r–neighbourhood of $M \subset L^\infty(M)$.

Definition 24 (Filling Radius, [39])**.** The *filling radius* of M, denoted by $\mathrm{FillRad}(M)$, is the infimum of those $r > 0$ for which $\iota_{r,*}([M]) = 0$, where $[M]$ is the fundamental class of M.

In his seminal paper [39], Gromov stated two interesting observations related to the construction of Definition 24. First, the filling radius decreases under distance-decreasing maps. This is due to a universal property of $L^\infty(M)$: an arbitrary distance-decreasing map of a subspace of a metric space into $L^\infty(M)$,

$$Y \to L^\infty(M) \text{ for } Y \subset X, \tag{2.1}$$

extends to a distance-decreasing map $X \to L^\infty(M)$. We can extend $y \to f_y(\cdot) \in L^\infty(M)$ by the following function:

$$x \to f_x(\cdot) = \inf_{y\in Y} (f_y(\cdot) + \operatorname{dist}_X(x, y)),$$

for all $x \in X$. Moreover, every distance-decreasing map $X_1 \to X_2$ extends to a distance-decreasing map $L^\infty(X_1) \to L^\infty(X_2)$. As there exists a distance-decreasing map $X_1 \to X_2$, due to the Kuratowski embedding, we can build a distance-decreasing map $X_1 \to L^\infty(X_2)$. Furthermore, as $X_1 \subset L^\infty(X_1)$, due to the property explained above, we obtain the distance-decreasing application $L^\infty(X_1) \to L^\infty(X_2)$.

Thus, if $X_1 \to X_2$ is distance-decreasing and a degree one map (for more information about degree one maps we recommend [44]), we have that $[X_1] \to \pm[X_2]$, where $[\cdot]$ denotes the fundamental class. Now, if $r < \mathrm{FillRad}(X_2)$, we have that $\iota_{r,*}([X_2]) \neq 0$. Then by following the diagram,

$$\begin{tikzcd} {[X_1]} \arrow[r] \arrow[d] & {[X_2]} \arrow[d] \\ \iota_{r,*}([X_1]) \arrow[r] & \iota_{r,*}([X_2]) \end{tikzcd} \tag{2.2}$$

we also have that $\iota_{r,*}([X_1]) \neq 0$ and then

$$\mathrm{FillRad}(X_1) \geq \mathrm{FillRad}(X_2).$$

Secondly, the construction of the filling radius could be done whenever an isometric embedding of the manifold exists but Gromov, indeed, proved that the one provided by the Kuratowski embedding gives the minimum value of any other filling radius. Using the same extension property as in the proof of the above observation, we can construct a distance-decreasing map $M \hookrightarrow X \to L^\infty(M)$. Now, as we have an isometry in (2.1) due to the Kuratowski embedding, using the same argument as in diagram (2.2), we obtain that

$$\mathrm{FillRad}(M \subset X) \geq \mathrm{FillRad}(M \subset L^\infty(M)).$$

Although the Filling Radius seems difficult to compute, Katz [53, 54, 55] obtained some values for it for some manifolds:

1. $$\mathrm{FillRad}(\mathbb{S}^n) = \frac{1}{2}\arccos\left(-\frac{1}{n+1}\right),$$

   where we endow $\mathbb{S}^n$ with the canonical metric of constant curvature 1.

2. $$\mathrm{FillRad}(\mathbb{R}P^n) = \frac{1}{3}\,\mathrm{diam}(\mathbb{R}P^n) = \frac{\pi}{6},$$

   where we endow $\mathbb{R}P^n$ with the metric of constant curvature 1.

3. $$\mathrm{FillRad}(\mathbb{C}P^1) = \mathrm{FillRad}(\mathbb{C}P^2) = \frac{1}{2}\arccos\left(-\frac{1}{3}\right),$$

   where we endow $\mathbb{C}P^1$ and $\mathbb{C}P^2$ with a metric with curvature $\frac{1}{4} \leq K \leq 1$.

4. $$\mathrm{FillRad}(\mathbb{C}P^n, \mathbb{Q}) = \frac{1}{2}\arccos\left(-\frac{1}{3}\right),$$

   where the field used to compute the homology is $\mathbb{Q}$ and $\mathbb{C}P^n$ has the canonical metric with curvature $\frac{1}{4} \leq K \leq 1$.

We also have the following inequalities [54]:

1. $$\mathrm{FillRad}(\mathbb{C}P^n) \geq \frac{1}{2}\arccos\left(-\frac{1}{3}\right).$$

2.

$$\mathrm{FillRad}(CaP^2) \geq \frac{1}{2}\arccos\left(-\frac{1}{9}\right).$$

3.

$$\mathrm{FillRad}(\mathbb{H}P^n) \geq \frac{1}{2}\arccos\left(-\frac{1}{5}\right).$$

The lack of explicit computations for the general case shows how difficult working with this invariant is.

There are some other important inequalities regarding the Filling Radius. The most important one is due to Gromov [39] and involves the Riemannian volume of an n–manifold and a constant depending on the dimension:

Theorem 2.1 ([39])**.** *Let V be a closed Riemannian connected Riemannian n–manifold. Then*

$$\mathrm{FillRad}(V) \leq C(n)(\mathrm{vol}(V))^{1/n},$$

for some universal constant

$$0 < C(n) < (n+1)n^n\sqrt{n!}.$$

Another important result is the one obtained by Katz [54] bounding any closed Riemannian manifold M by its diameter and the *spread*:

Theorem 2.2 ([54])**.** *Let M be a closed Riemannian manifold, if its Filling Radius exists, then*

$$\mathrm{FillRad}(M) \leq \frac{1}{3}\mathrm{diam}(M).$$

Moreover,

$$\mathrm{FillRad}(M) \leq \frac{1}{2}\mathrm{Spread}(M),$$

where the $\mathrm{Spread}(M)$ *is the smallest $R > 0$ so that there is a closed subset $Y \subset M$ with* $\mathrm{diam}(Y) \leq R$ *and* $\mathrm{dist}_M(x, Y) \leq R$ *for every $x \in M$.*

The Filling Radius has also been studied under curvature assumptions on M. In that sense, Wilhelm and Yokota [93, 95] compared the filling radius of a positively curved space to that of the unit n–sphere.

Theorem 2.3 (Main Theorem 2, [93])**.** *Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$, and let $\mathcal{M}$ denote the class of closed, Riemannian n–manifolds with sectional curvature ≥ 1. For all $M \in \mathcal{M}$,*

1. $\mathrm{FillRad}(M) \leq \mathrm{FillRad}(\mathbb{S}^n)$.
2. *If* $\mathrm{FillRad}(M) = \mathrm{FillRad}(\mathbb{S}^n)$*, then M is isometric to $\mathbb{S}^n$.*
3. *There is a $\delta(n) > 0$ so that if* $\mathrm{FillRad}(\mathbb{S}^n) - \delta(n) > \mathrm{FillRad}(M)$*, then M is diffeomorphic to $\mathbb{S}^n$.*
4. *If* $\mathrm{FillRad}(M) > \frac{\pi}{6}$ *then M is a twisted sphere.*

In order to present the result for Alexandrov spaces we need to define the *contractibility radius* $\operatorname{Cont}_k \operatorname{Rad}(X)$ of a metric space X. Firstly, we introduce the concept of *k–degenerate* map:

Definition 25 (k–degenerate map, [39]). A continuous map $f\colon X \to Y$ between metric spaces is sad to be *k–degenerate* if it factors as $f = f' \circ f''$ through a k–dimensional polyhedron K for some continuous maps $f'\colon K \to Y$ and $f''\colon X \to K$.

Remark. With the definition of k–degenerate maps and

$$\operatorname{Diam} f = \sup_{x_1, x_2 \in X} [\operatorname{dist}_X(x_1, x_2) - \operatorname{dist}_Y(f(x_1), f(x_2))],$$

for $f\colon X \to Y$ between metric spaces, Gromov also defines in [39] the *k–diameter* of X:

$$\operatorname{Diam}_k X = \inf_{f,Y} \operatorname{Diam} f.$$

Definition 26 (Contractibility radius, [39]). For an arbitrary compact subspace V in a metric space X we define the *contractibility radius* $\operatorname{Cont}_k \operatorname{Rad}(V \subset X)$ to be the lower bound of $\epsilon > 0$, for which the inclusion map of V into its ϵ–neighbourhood $V \subset U_\epsilon(V)$ is a k–contractible map, i.e., a continuous map between two metric spaces that is homotopic to a k–degenerate map.

Theorem 2.4 (Corollary 8, [96]). *For any n–dimensional Alexandrov space X of curvature ≥ 1 with $\partial X = \emptyset$, either $\operatorname{FillRad}(X) < \operatorname{FillRad}(\mathbb{S}^n)$ or X is isometric to the round sphere $\mathbb{S}^n$.*

Moreover, for any n–dimensional Alexandrov space X of curvature ≥ 1, either

$$\operatorname{Cont}_k \operatorname{Rad}(X) < \operatorname{Cont}_k \operatorname{Rad}(\mathbb{S}^n) = l_n/2$$

for any $0 \leq k \leq n-1$ or X is isometric to the round sphere $\mathbb{S}^n$.

The Filling Radius is such an interesting invariant that even a foreign field as Topological Data Analysis (TDA) has tried to used it in order to obtain results. In this case, we can see how Mémoli et. al on [63] used it in order to extract properties of the persistence diagram, PD_n (recall that a definition of a persistence diagram can be found on Section 1.5.3), of the highest dimension:

Proposition 2.2 ([63], Proposition 9.4). *Let M be a closed connected Riemannian n–manifold. Then,*

$$(0, 2 \operatorname{FillRad}(M)) \in PD_n(VR(M); \mathbb{F}),$$

where $VR(M)$ denotes the Vietoris–Rips construction over M and $\mathbb{F}$ is any field if M is orientable and $\mathbb{F} = \mathbb{Z}_2$ if M is non-orientable. Moreover, this is the unique point in $PD_n(VR(M))$ with x–coordinate equal to 0.

They also proved an interesting result concerning the stability of the filling radius:

Proposition 2.3 ([63], Proposition 9.10). *Let M be a closed connected n–dimensional manifold. Let dist^1_M and dist^2_M be a two metrics on M compatible with the manifold topology. Then,*

$$|\operatorname{FillRad}(M, \operatorname{dist}^1_M) - \operatorname{FillRad}(M, \operatorname{dist}^2_M)| \leq || \operatorname{dist}^1_M - \operatorname{dist}^2_M ||_\infty.$$

2.3 Positiveness of the Filling Radius

In this section, we give a lower bound for the Filling Radius of an arbitrary closed Riemannian manifold in terms of its injectivity radius and an upper curvature bound.

The following result, a particular case of the main Theorem in [1], will be of particular importance.

Theorem 2.5 ([1], Main Theorem)**.** *Let (M, g) be a complete Riemannian manifold with* $\sec \leq K$, *and ν a probability measure in M such that its support is contained in a ball of radius ρ where*

$$\rho < \frac{1}{2} \min \left\{ \operatorname{inj} M \, , \, \frac{\pi}{\sqrt{K}} \right\}.$$

Then the function $F_2 : M \to \mathbb{R}$ defined as $F_2(q) = \frac{1}{2} \int_M \operatorname{dist}^2(p, q) \, d\nu(p)$ has a unique minimizer.

A similar statement appears in [37, Remark on Section 3]. Greene and Petersen showed the positiveness of the Filling Radius in terms of the convexity radius of the space. As far as we know, this is the first extended proof for this estimation:

Theorem 2.6 (C. & Guijarro, [31])**.** *Let M be a closed Riemannian manifold with injectivity radius* $\operatorname{inj} M$ *and with sectional curvature* $\sec \leq K$*, where $K \geq 0$. Then*

$$\operatorname{FillRad}(M) \geq \frac{1}{4} \min \left\{ \operatorname{inj} M \, , \, \frac{\pi}{\sqrt{K}} \right\}, \tag{2.3}$$

where $\pi/\sqrt{K}$ is understood as ∞ whenever $K = 0$.

Proof. We will show that there is a continuous retraction of the tubular neighbourhood $\Phi : U_R(M) \to M$, by associating to each function $f \in U_R(M)$ the center of mass of a set in M associated to f (see [40], [51], [1]). To improve readability, we will denote the right hand side in (2.3) by R_0. For each $R > 0$, denote by $U_R(M)$ the open R-neighbourhood of M inside $L^\infty(M)$, i.e.,

$$U_R(M) = \{ f \in L^\infty(M) \, : \, \operatorname{dist}(f, M) < R \} .$$

Whenever $f \in U_R$, we define the *vicinity set of f* as

$$A_f^R := \{ p \in M \, : \, \| f - \operatorname{dist}_p \| \leq R \} .$$

It is clear from the definition that the sets A_f^R are closed with nonempty interior, and for $R \leq R'$, there is an inclusion $A_f^R \subset A_f^{R'}$; we also have that if $p, q \in A_f^R$, then

$$\operatorname{dist}(p, q) = || \operatorname{dist}_p - \operatorname{dist}_q ||_\infty \leq 2R,$$

by the triangle inequality, thus $\operatorname{diam} A_f^R \leq 2R$. Furthermore, if $p \in A_f^R$, and $g \in L^\infty(M)$, then

$$|| \operatorname{dist}_p - g||_\infty \leq ||f - g||_\infty + || \operatorname{dist}_p - f||_\infty,$$

and therefore $p \in A_g^{R+||g-f||_\infty}$. Interchanging the roles of f and g, we get the sequence of inclusions

$$A_f^R \subset A_g^{R+||g-f||_\infty} \subset A_f^{R+2||g-f||_\infty}.$$

Notice also that, as $\varepsilon \to 0$, we get that

$$A_f^{R+\varepsilon} \to A_f^R, \quad \text{that is, } \bigcap_{\varepsilon>0} A_f^{R+\varepsilon} = A_f^R.$$

We will denote the characteristic function of a set A as χ_A and the Riemannian measure of M as $d\,\mathrm{vol}$. If we have a sequence $f_n \to f$ in $L^\infty(M)$, we have that, writing $\varepsilon_n = ||f_n - f||$, we obtain

$$\int_M |\chi_{A^R_{f_n}} - \chi_{A^R_f}|\, d\,\mathrm{vol} \le \int_M |\chi_{A^R_{f_n}} - \chi_{A^{R+\varepsilon_n}_f}|\, d\,\mathrm{vol} + \int_M |\chi_{A^{R+\varepsilon_n}_f} - \chi_{A^R_f}|\, d\,\mathrm{vol},$$

and thus

$$\int_M |\chi_{A^R_{f_n}} - \chi_{A^R_f}|\, d\,\mathrm{vol} \to 0, \tag{2.4}$$

as $n \to \infty$.

The set A_f^R has nonempty interior, since it contains the set of points with $||f - \mathrm{dist}_q|| < R$, thus its volume in M does not vanish. We can then consider the probability measure in M defined as

$$\nu_f = \frac{1}{\mathrm{vol}(A_f)} \chi_{A_f}\, d\,\mathrm{vol},$$

where vol is the Riemannian volume and χ_{A_f} is the characteristic function of the set A_f. Its support is A_f, and thus it has diameter less or equal than $2R$. Observe that as $f_n \to f$ in L^∞, the characteristic functions of A_{f_n} converge to the characteristic function of A_f, and consequently, for any continuous function $g : M \to \mathbb{R}$,

$$\int_M g(x) d\nu_{f_n} \to \int_M g(x) d\nu_f.$$

Let $0 < R < R_0$, where R_0 was defined as the right hand side in inequality (2.3). For any $f \in U_R$, its vicinity set is contained in a ball of radius $2R$, thus the main result in [1, Theorem 2.1] can be applied to the measure ν_f to obtain a unique point $p \in M$ characterized as the single minimizer of the function

$$F_2^f : M \to \mathbb{R}, \qquad F_2^f(q) := \frac{1}{2}\int_M d^2(p,q)\, d\nu_f(p).$$

From equation (2.4), it is clear that the assignment $\Phi : U_R(M) \to M$ mapping f to p is continuous. Moreover, when $f = \mathrm{dist}_p$, $A_f^R = B_R(p)$, and the minimum of F_2^f agrees with p, thus Φ is a retraction, and

$$M \xrightarrow{\iota_{R_0}} U_{R_0}(M) \xrightarrow{\Phi} M$$

is the identity map and $(\Phi \circ \iota_{R_0})_\#[M] = [M]$. However, if $R_0 > \mathrm{FillRad}(M)$, $(\iota_{R_0})_\#[M] = 0$. This finishes the proof. $\square$

Corollary 2.6.1. *Let (M, g) be a closed Riemannian manifold, then*

$$\mathrm{FillRad}(M) \ge c_o > 0.$$

2.4 Filling Radius bounds of Riemannian submersions

On [31], we obtained a lower bound for the total space M of a Riemannian submersion in terms of the diameter of the fibers:

Theorem 2.7 (C. & Guijarro, [31]). *Let $\pi : M \to B$ be a Riemannian submersion with $\dim M > \dim B$. Then*

$$\operatorname{FillRad}(M) \leq \frac{1}{2} \max_{b \in B} \{\operatorname{diam} \pi^{-1}(b)\}, \tag{2.5}$$

where the diameter of each fiber is considered in the extrinsic metric.

Proof. To facilitate the writing, denote by ρ_0 the right hand side in (2.5). As π is a Riemannian submersion, we can isometrically embed $B \hookrightarrow L^\infty(M)$ with the map

$$\varphi_1 : B \to L^\infty(M), \qquad b \to \operatorname{dist}_b(\pi(\cdot)), \tag{2.6}$$

where $\operatorname{dist}_b(\pi(z)) := \operatorname{dist}_B(b, \pi(z))$, for all $z \in M$.

Next, we translate this embedding by adding ρ_0, that is, we have an isometric embedding

$$\varphi : B \to L^\infty(M), \qquad b \to f_b,$$

where

$$f_b : M \to \mathbb{R}, \qquad f_b(z) := \operatorname{dist}_b(\pi(z)) + \rho_0.$$

We will construct a deformation retract of M onto the image $\varphi(B)$ adapting the main idea in [54, Lemma 1] to our situation. The difference will be that, instead of constructing a cone over M in a tubular neighbourhood of M, we will construct a *mapping cylinder* of the Riemannian submersion $\pi : M \to B$. For this, define ρ_0 as the upper bound in (2.5), i.e,

$$\rho_0 := \frac{1}{2} \max_{b \in B} \{\operatorname{diam} \pi^{-1}(b)\},$$

where the diameter of each fiber is computed as

$$\operatorname{diam} \pi^{-1}(b) := \max\{ \operatorname{dist}_x(y) \ : \ \pi(x) = \pi(y) = b \},$$

and for each $t \in [0, \rho_0]$, and for each $p \in \pi^{-1}(b)$, construct the function

$$\operatorname{dist}_p^t(z) = \begin{cases} \min\{\operatorname{dist}_p(z) + t, f_b(z)\}, & \text{if } \operatorname{dist}_p(z) < f_b(z) \\ \max\{\operatorname{dist}_p(z) - t, f_b(z)\}, & \text{if } \operatorname{dist}_p(z) \geq f_b(z). \end{cases}$$

We start by proving that $|| \operatorname{dist}_p^t - \operatorname{dist}_p ||_\infty \leq t$. For this, we consider two cases:

1. When $\operatorname{dist}_p(z) < f_b(z)$, the function dist_p^t is given by $\min\{\operatorname{dist}_p(z) + t, f_b(z)\}$; then we need to estimate

$$\max_{\{z \in M \,:\, \operatorname{dist}_p(z) < f_b(z)\}} |\min\{\operatorname{dist}_p(z) + t, f_b(z)\} - \operatorname{dist}_p(z)|.$$

For this, we are going to use the sets:

$$\begin{aligned} A &= \{z : \operatorname{dist}_p(z) < f_b(z)\}, \\ B &= \{z : \operatorname{dist}_p(z) + t > f_b(z)\}, \\ B^c &= \{z : \operatorname{dist}_p(z) + t \leq f_b(z)\}. \end{aligned}$$

If $z \in A \cap B^c$, we have that

$$\max_{z\in A\cap B^c} |\operatorname{dist}_p^t(z) - \operatorname{dist}_p(z)| = \max_{z\in A\cap B^c} |\operatorname{dist}_p(z) + t - \operatorname{dist}_p(z)| = t.$$

On the other hand,

$$A \cap B = \{z : \operatorname{dist}_p(z) < f_b(z) < \operatorname{dist}_p(z) + t\} = \{z : 0 < f_b(z) - \operatorname{dist}_p(z) < t\},$$

so, if $z \in A \cap B$, we have that

$$\max_{z\in A\cap B} |\operatorname{dist}_p^t(z) - \operatorname{dist}_p(z)| = \max_{z\in A\cap B} |f_b(z) - \operatorname{dist}_p(z)| < t.$$

Therefore, on A we have obtained that

$$|\operatorname{dist}_p^t - \operatorname{dist}_p| \leq t.$$

2. Suppose next that $z \in A^c$, i.e., $\operatorname{dist}_p(z) \geq f_b(z)$. Now, we have to compute

$$\max_{\{z\in M\,:\,\operatorname{dist}_p(z)\geq f_b(z)\}} |\max\{\operatorname{dist}_p(z) - t, f_b(z)\} - \operatorname{dist}_p(z)|.$$

In this case, we take the partition of M given by the sets

$$\begin{aligned} C &= \{z : \operatorname{dist}_p(z) - t > f_b(z)\}. \\ C^c &= \{z : \operatorname{dist}_p(t) - t \leq f_b(z)\}. \end{aligned}$$

If $z \in A^c \cap C$, we have that

$$\max_{z\in A^c\cap C} |\operatorname{dist}_p^t(z) - \operatorname{dist}_p(z)| = \max_{z\in A^c\cap C} |\operatorname{dist}_p(z) - t - \operatorname{dist}_p(z)| = t.$$

On the other hand,

$$A^c \cap C^c = \{z : \operatorname{dist}_p(z) \geq f_b(z) \geq \operatorname{dist}_p(z) - t\} = \{z : 0 \leq \operatorname{dist}_p(z) - f_b(z) \leq t\},$$

and

$$\max_{z\in A^c\cap C^c} |\operatorname{dist}_p^t(z) - \operatorname{dist}_p(z)| = \max_{z\in A^c\cap C^c} |f_b(z) - \operatorname{dist}_p(z)| \leq t.$$

Thus, on A^c we have that $|\operatorname{dist}_p^t - \operatorname{dist}_p| \leq t$, and combining it with the above, we have finally obtained that

$$||\operatorname{dist}_p^t - \operatorname{dist}_p||_\infty \leq t.$$

We prove next that for ρ_0 as defined as the upper bound in (2.5), we have that

$$\operatorname{dist}_p^{\rho_0} = f_b, \qquad \text{where } b = \pi(p).$$

Once again, we divide into cases:

1. If $z \in A$, then our claim is clear, since for any Riemannian submersion,

$$\mathrm{dist}(p, z) \geq \mathrm{dist}_B(\pi(p), \pi(z)),$$

and thus

$$f_b(z) = \mathrm{dist}_B(\pi(p), \pi(z)) + \rho_0 \leq \mathrm{dist}(p, z) + \rho_0.$$

2. On the other hand, for any $z \in M$, we have that there is at least one point q in the fiber through p with $\mathrm{dist}(z, q) = \mathrm{dist}_B(\pi(z), \pi(p))$, thus (see Figure 1)

$$\mathrm{dist}(p, z) \leq \mathrm{dist}(p, q) + \mathrm{dist}(q, z) \leq \mathrm{diam}\, \pi^{-1}(b) + \mathrm{dist}_B(\pi(z), \pi(p)) \leq f_b(z) + \rho_0.$$

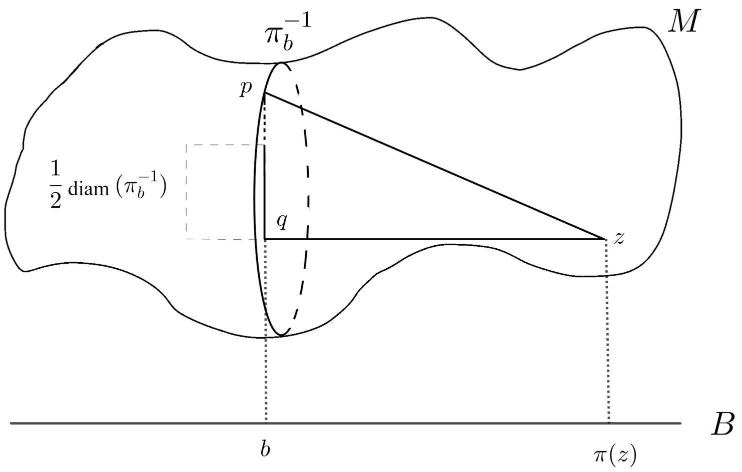

Figure 2.1:

If Cyl_π is the cylinder map of $\pi : M \to B$, the map dist_p^t induces a map

$$\psi : \mathrm{Cyl}_\pi \to L^\infty(M), \qquad [p, t] \to \mathrm{dist}_p^t(\cdot)$$

where $\psi(p, 0)$ agrees with the Kuratowski embedding of M, and $\psi(M, \rho_0)$ agrees with the image of the inclusion of B in $L^\infty(M)$ given by $b \to f_b$. Moreover, the image of ψ is contained in the tubular neighbourhood of radius ρ_0 around M by the above computations. Since Cyl_π retracts onto $\psi(M, \rho_0)$, and $\dim B < \dim M$, the image of the fundamental class of M vanishes in U_{ρ_0}, finishing the proof.

□

There is a better estimate for Riemannian products, since Gromov proved in [39] that the filling radius of a product satisfies $\mathrm{FillRad}(M_1 \times M_2) = \min(\mathrm{FillRad}(M_1), \mathrm{FillRad}(M_2))$. For warped products our theorem provides:

Corollary 2.7.1. *For B, F closed Riemannian manifolds, let $f : B \to (0, \infty)$ be a smooth function, and $M = B \times_f F$ the warped product over B with fiber F. Then*

$$\mathrm{FillRad}(M) \leq \min\{\, \mathrm{FillRad}(B), \frac{1}{2} \max f \cdot \mathrm{diam}\, F \,\}.$$

Proof. $\mathrm{FillRad}(M)$ can not exceed half the maximum diameter of the fibers by the previous theorem; by the definition of warped product, this explains the second term appearing in the above minimum. $\mathrm{FillRad}(B)$ appears by exactly the same explanation as in [39, Pages 8-9] □

2.5 Filling radius of submetries

As it is remarked in [63], the definition of filling radius does not require the distance of M to come from a Riemannian metric. It suffices that $\widehat{\mathrm{dist}}_M$ generates the manifold topology. We call any $(M, \widehat{\mathrm{dist}}_M)$ satisfying this condition a *metric manifold*.

When going over the proof of Theorem 2.7, it is clear that the proof is entirely metric, and, except for the total space needing enough structure to have a fundamental class, the rest of the arguments carry verbatim to provide the following result:

Corollary 2.7.2. *Let $(X, \widehat{\mathrm{dist}}_X)$ be a metric manifold (i.e, a closed manifold with a distance), (Y, dist_Y) a metric space and $\pi : X \to Y$ a submetry between them. Thus*

$$\mathrm{FillRad}(X) \leq \frac{1}{2} \max_{y \in Y} \{\mathrm{diam}\, \pi^{-1}(y)\}.$$

We should mention that, in the above corollary, X does not need to be a manifold at all; for instance, X can be replaced by a closed Alexandrov space, since such spaces have fundamental classes by the work of Yamaguchi in [94].

This corollary has some useful consequences that extend Theorem 2.7:

Corollary 2.7.3. *Suppose M is a Riemannian manifold admitting a singular Riemannian foliation $\mathcal{F}$ with closed leaves. Then*

$$\mathrm{FillRad}(M) \leq \frac{1}{2} \max_{N \in \mathcal{F}} \{\mathrm{diam}\, N\}.$$

As an example of the above, recall that when G is a compact Lie group acting by isometries on a closed Riemannian manifold M, the orbits of G form a singular Riemannian foliation in M, thus the filling radius of M cannot exceed one half the diameter of the orbits.

Example 10. We can give an example where the bound in Corollary 2.7.2 is better than the one–third–diam bound. Let M^n be a cohomogeneity one manifold with Grove-Ziller diagram $H \subset K = K' \subset G$ (see [41, 46]). Then M^n can be given a cohomogeneity one metric containing a product cylinder $G/H \times [-\ell, \ell]$ with ℓ as large as desired. Its diameter will exceed 2ℓ, while the diameter of the fibers of M/G remains uniformly bounded above.

2.6 Intermediate Filling Radius

If, instead of the fundamental class of the manifold, we look at a different fixed homology class in M, we arrive at the concept of the intermediate filling radius introduced in [63]. We adapt their definition to the case of $M \subset L^\infty(M)$.

Definition 27 (Intermediate Filling Radius)**.** For any integer $k \geq 1$, any abelian group $\mathbb{F}$, and any homology class $\omega \in \mathrm{H}_k(M;\mathbb{F})$, we define the *$k$-intermediate filling radius of ω* as

$$\mathrm{FillRad}_k(M,\mathbb{F},\omega) := \inf\{r > 0 : \iota_{r,*}(\omega) = 0\},$$

where $\iota_r : M \hookrightarrow U_r(M)$ is the isometric embedding. This gives us a map

$$\mathrm{FillRad}_k(M,\mathbb{F},\cdot) : \mathrm{H}_k(M;\mathbb{F}) \to \mathbb{R}_{\geq 0}.$$

Finally, the *intermediate k-filling radius of M* is the infimum of the function $\mathrm{FillRad}_k(M,\mathbb{F},\cdot)$ over $\mathrm{H}_k(M;\mathbb{F})$.

We will usually omit $\mathbb{F}$ from the notation if it does not create confusion. Our first observation is to restrict the possible values of $\mathrm{FillRad}_k$.

Proposition 2.4. *For M as in Theorem 2.6, we have that*

$$\mathrm{FillRad}_k(M) \geq \frac{1}{4}\left\{\mathrm{inj}\, M\,,\ \frac{\pi}{\sqrt{K}}\right\}. \tag{2.7}$$

Proof. For R smaller than the right hand side in (2.7), we have a retraction of $U_R(M)$ onto M, as proven in Theorem 2.6. □

Now, we present a Theorem stated by Liu in [64], that we are going to generalize in our next result:

Theorem 2.8 ([64], Theorem 1.1)**.** *Let V, W be closed, connected, oriented Riemannian n–manifolds. Suppose $f\colon V \to W$ has nonzero degree. Then*

1. *if $|\deg(f)| = 1$, $\mathrm{FillRad}(W) \leq C \cdot \mathrm{FillRad}(V)$.*
2. *if $|\deg(f)| > 1$, $\mathrm{FillRad}(W,\mathbb{Q}) \leq C \cdot \mathrm{FillRad}(V)$,*

where

$$C = \mathrm{dil}\, f = \sup_{p \neq p'} \frac{\mathrm{dist}_N(f(p), f(p'))}{\mathrm{dist}_M(p,p')}.$$

The following theorem can be considered a mild extension of the previous one by Liu to this intermediate invariant.

Theorem 2.9. *Let $f : M^m \to N^n$ be a Lipschitz map between closed manifolds such that the induced map $f_{k,*} : \mathrm{H}_k(M) \to \mathrm{H}_k(N)$ is onto. Then*

$$\mathrm{FillRad}_k(M) \geq C^{-1}\,\mathrm{FillRad}_k(N).$$

Proof. As in [64], we extend the map $f : M \to N \to L^\infty(N)$ to a map $\widetilde{f} : L^\infty(M) \to L^\infty(N)$ that remains Lipschitz with the same dilation as f. Then

$$\widetilde{f}(U_R(M)) \subset U_{C\cdot R}(N),$$

and we have a commutative diagram in homology

$$\begin{tikzcd} \mathrm{H}_k(M) \arrow[r, "f_{k,*}"] \arrow[d, "\iota_*"'] & \mathrm{H}_k(N) \arrow[d, "\iota_*"] \\ \mathrm{H}_k(U_R(M)) \arrow[r, "\widetilde{f}_{k,*}"] & \mathrm{H}_k(U_{C\cdot R}(N)) \end{tikzcd}$$

If $C \cdot R < \mathrm{FillRad}_k(N)$, there is some $a \in \mathrm{H}_k(N)$ such that $\iota_*(a) \neq 0$, hence there is some $b \in \mathrm{H}_k(M)$ such that $\widetilde{f}_{k,*}\iota_*(b) \neq 0$; then $\iota_*(b) \neq 0$, and $R < \mathrm{FillRad}_k(M)$.

□

Chapter 3

On the reach of isometric embeddings of metric spaces

Recall that, as we explained in Section 1.6, the reach of a subset measures how much the subset folds in on itself (i.e., how close apart two pieces of the set are in the ambient space despite them being far in the intrinsic metric of the set).

For the clarity of this chapter, we recall the definition of the set of *unique points* and *reach* of a subset:

Definition 28 (Unique points set and reach, [32])**.** Let (X, dist) be a metric space and $A \subset X$ a subset. We define the set of points having a unique metric projection in A as

$$\mathrm{Unp}(A) = \{x \in X : \text{there exists a unique } a \text{ such that } \mathrm{dist}(x, A) = \mathrm{dist}(x, a)\}.$$

For $a \in A$, we define the *reach* of A at a, denoted by $\mathrm{reach}(a, A)$, as

$$\mathrm{reach}(a, A) = \sup\{r \geq 0 : B_r(a) \subset \mathrm{Unp}(A)\}.$$

Finally, we define the *global reach* by

$$\mathrm{reach}(A) = \inf_{a \in A} \mathrm{reach}(a, A).$$

In this chapter, we study the value of the reach of four isometric embeddings: that of the Kuratowski embedding, and into three Wasserstein-type spaces (canonical Wassertein, Orlicz–Wassertein and the space of persistence diagrams). All these results appear in the following articles: Manuel Cuerno and Luis Guijarro. “Upper and lower bounds on the filling radius”. In: *Indiana Univ. Math. J.* (2022). URL: https://arxiv.org/abs/2206.08032. Forthcoming, Javier Casado, Manuel Cuerno, and Jaime Santos-Rodríguez. *On the reach of isometric embeddings into Wasserstein type spaces*. 2023. arXiv: 2307.01051.

3.1 Reach of the Kuratowski embedding

The final part of [31] deals with the computation of the reach of the Kuratowski embedding into $L^\infty(M)$.

Theorem 3.1 ([31]). *Let M^n be a compact Riemannian manifold. For every $p \in M$,*

$$\operatorname{reach}(p, M \subset L^\infty(M)) = 0.$$

Proof. We need to prove that, given an arbitrary $\epsilon > 0$, and $p \in M$, there is some function $f \in B_\epsilon(p)$ and a point $q \neq p \in M$ such that

$$\operatorname{dist}_\infty(f, M) = || \operatorname{dist}_p - f||_\infty = || \operatorname{dist}_q - f||_\infty. \tag{3.1}$$

Let $0 < \delta << \epsilon$, and choose $q \in M$ such that

$$|| \operatorname{dist}_p - \operatorname{dist}_q ||_\infty = \operatorname{dist}_M(p, q) = \delta.$$

Define

$$f : M \to \mathbb{R}, \qquad f(s) := \frac{1}{2}(\operatorname{dist}_p(s) + \operatorname{dist}_q(s)).$$

We claim that this function does not belong to the Kuratowski embedding of M: otherwise, there would be some $r \in M$ with

$$f(s) = \frac{1}{2}(\operatorname{dist}_p(s) + \operatorname{dist}_q(s)) = \operatorname{dist}_r(s),$$

and evaluating at $s = r$, we would obtain that $\operatorname{dist}_p(r) = 0 = \operatorname{dist}_q(r)$, and p, q and r would all be the same point. Also, f lies in $B_\epsilon(p)$ since

$$||f - \operatorname{dist}_p ||_\infty = ||\frac{1}{2}(\operatorname{dist}_p + \operatorname{dist}_q) - \operatorname{dist}_p ||_\infty = \frac{1}{2}|| \operatorname{dist}_q - \operatorname{dist}_p ||_\infty = \frac{\delta}{2} < \epsilon.$$

Moreover, a similar computation yields that

$$||f - \operatorname{dist}_q ||_\infty = \frac{\delta}{2} = ||f - \operatorname{dist}_p ||_\infty.$$

It only remains to show that there is not any point in M closer to f. So assume that there were some $\operatorname{dist}_r \in M$ such that

$$|| \operatorname{dist}_r - f||_\infty < || \operatorname{dist}_p - f||_\infty.$$

Then

$$|| \operatorname{dist}_r - f||_\infty < || \operatorname{dist}_p - f||_\infty = \frac{\delta}{2}.$$

Evaluating at r, we would obtain

$$| \operatorname{dist}_r(r) - f(r)| = |f(r)| = \frac{1}{2}\left(\operatorname{dist}_p(r) + \operatorname{dist}_q(r)\right) < \frac{\delta}{2}. \tag{3.2}$$

On the other hand, by the triangle inequality,

$$\operatorname{dist}_p(r) + \operatorname{dist}_q(r) \geq \operatorname{dist}_M(p, q) = \delta \tag{3.3}$$

resulting in a contradiction. Thus f satisfies (3.1) and the proof is finished.

□

All this study of the Kuratowski embedding and the reach led us to some open questions that we have not solved yet:

Question 1. Is $\operatorname{Unp}(M \subset L^\infty(M))$ dense in $L^\infty(M)$?

Question 2. Give a characterization of $\operatorname{Unp}(M \subset L^\infty(M))$.

3.2 Reach of the Wasserstein space

Before presenting the results concerning the reach in the Wasserstein space, as we are going to use this concept in the rest of the chapter, we want to introduce the definition of *p–barycenter*:

Definition 29 (p–barycenter)**.** We define a *p–barycenter* of a measure $\mu \in W_p(X)$, where (X, dist_X) is a metric space, as a point in X that minimizes the distance between μ and some isometric embedding of the metric space inside that space, in particular, the canonical embedding that sends $x \mapsto \delta_x$.

Remark. As we are going to deal with other Wasserstein type spaces, we will naturally extend the definition of p–barycenter in each context.

The first open question for the Unp set of the Kuratowski embedding deals with the density of this subset. We have solved this question in the canonical Wasserstein space case: The set of unique points of the isometric embedding of a metric space into its p–Wasserstein space is dense in the total space:

Proposition 3.1 ([24])**.** *Let (X, dist) be a non-branching metric space and $W_p(X)$ with $p > 1$ its p–Wasserstein space. Then the set of unique points $\mathrm{Unp}(X \subset W_p(X))$ is dense in $W_p(X)$.*

Proof. Let $\mu \in W_p(X)$ be a measure with a barycenter $x \in X$. Take ν inside a geodesic between δ_x and μ. Suppose that there exists some other point $z \in X$ that is a barycenter for ν. This implies that $W_p(\nu, \delta_z) \leq W_p(\nu, \delta_x)$ and with this we get

$$W_p(\mu, \delta_z) \leq W_p(\mu, \nu) + W_p(\nu, \delta_z) \leq W_p(\mu, \nu) + W_p(\nu, \delta_x) = W_p(\mu, \delta_x).$$

Hence z is also a barycenter for μ. Furthermore, we notice that there is a branching geodesic joining μ with δ_z as at some point it branches at ν in order to also join μ and δ_x. This gives us the contradiction as $W_p(X)$ is non-branching. Then ν is a measure in $\mathrm{Unp}(X \subset W_p(X))$ which can be taken arbitrarily close to μ. $\square$

As $\mathrm{Unp}(X \subset W_p(X))$ is dense in $W_p(X)$, a natural question about the positivity of the reach in this context seems natural: Is there any metric space (X, dist_X) with

$$\mathrm{reach}(x, X \subset W_p(X)) > 0$$

for all $x \in X$?

3.2.1 Null reach

The first result concerning the reach and the Wasserstein space is that the reach of a metric space inside its 1–Wasserstein space is always 0. The proof follows the idea of the proof of Theorem 3.1.

Theorem 3.2 ([24])**.** *Let (X, dist) be a metric space, and consider its 1–Wasserstein space, $W_1(X)$. Then, for every accumulation point $x \in X$,*

$$\mathrm{reach}(x, X \subset W_1(X)) = 0.$$

In particular, if X is not discrete, $\mathrm{reach}(X \subset W_1(X)) = 0$.

Proof. We are going to follow the same spirit as in the proof of Theorem 3.1.

Let $\epsilon > 0$. We will show that inside $B_\epsilon(x) \subset W_1(X)$ there exists at least one measure $\mu \notin \mathrm{Unp}(X)$.

By hypothesis, there exists $y \in X$, $y \neq x$, such that $d(x, y) < \epsilon$. Then

$$\mu := \frac{1}{2}\delta_x + \frac{1}{2}\delta_y.$$

First, notice that $\mu \neq \delta_z$ for any $z \in X$ because the support of μ is different from the support of any of the $\delta_z \in X$. In addition, due to the triangle inequality,

$$W_1(\delta_a, \mu) = \frac{1}{2}\operatorname{dist}(a, x) + \frac{1}{2}\operatorname{dist}(a, y) \geq \frac{1}{2}\operatorname{dist}(x, y). \tag{3.4}$$

By inequality (3.4) above, we can clearly see that $\mu \in B_\epsilon(x)$, because

$$W_1(\delta_x, \mu) = \frac{1}{2}\operatorname{dist}(x, y) < \epsilon.$$

Finally, we observe that both $a = x$ and $a = y$ minimize the distance to μ. Therefore, $\mu \notin \mathrm{Unp}(X)$ and $\mathrm{reach}(x, X \subset W_1(X)) = 0$. □

Note that the hypothesis of the point being an accumulation point is necessary, because, if $x_0 \in X$ is an isolated point, then the quantity $\ell = \inf_{x \in X}\operatorname{dist}(x, x_0)$ is strictly positive, and $B_{\ell/2}(x)$ admits a unique metric projection to X as every $p \in B_{\ell/2}(x)$ is closer to x than any other point of X.

Remark. The accumulation point argument in the above proof can also be used in Theorem 3.1, so we can restate that theorem for accumulation points of compact metric spaces. Moreover, if the space is not discrete, the result holds for every point.

An interesting observation is that, combining the same argument in the proof of Theorem 3.2 with the previous remark, if X is a discrete metric space isometrically embedded into another metric space Y, then

$$\mathrm{reach}(X \subset Y) = \inf_{x_1 \neq x_2} \operatorname{dist}(x_1, x_2)/2 > 0.$$

Now we will provide results about the reach of a geodesic metric space inside its p–Wasserstein space with $p > 1$. We have found that these results are closely related to the uniqueness of the geodesics. This next proposition has important consequences about the reach inside a Wasserstein space, as it constructs measures with possibly several projections in X.

Proposition 3.2 ([24]). *Let (X, dist) be a geodesic metric space, and $x, y \in X$ two points with $x \neq y$. Consider the probability measure*

$$\mu = \lambda\delta_x + (1 - \lambda)\delta_y,$$

for $0 < \lambda < 1$. Then μ minimizes its p–Wasserstein distance to X exactly once for every minimizing geodesic between x and y.

Proof. The proof is structured in the following way: First, we choose a candidate for the distance–minimizer of μ, supposing it lies inside a minimizing geodesic. Then, we show that the global minimum distance can only be achieved inside a minimizing geodesic.

Choose $\gamma(t)\colon [0,1] \to X$ a minimizing geodesic from x to y. We can compute the cost $W_p^p(\delta_{\gamma(t)}, \mu)$ and then minimize in t. Indeed,

$$\begin{aligned} W_p^p(\delta_{\gamma(t)}, \mu) &= \lambda \operatorname{dist}(\gamma(t), x)^p + (1-\lambda) \operatorname{dist}(\gamma(t), y)^p \\ &= (\lambda t^p + (1-\lambda)(1-t)^p) \operatorname{dist}(x, y)^p \end{aligned}. \tag{3.5}$$

The minimum will be achieved at the parameter t_0 which verifies

$$\left.\frac{d}{dt}\right|_{t=t_0} W_p^p(\delta_{\gamma(t)}, \mu) = 0.$$

We know this because that derivative is negative for $t = 0$, and positive for $t = 1$, and vanishes at only one point $t = t_0$. An easy computation shows us that the only solution in our interval is

$$t_0 = \frac{(1-\lambda)^{p-1}}{\lambda^{p-1} + (1-\lambda)^{p-1}}.$$

Thus, the Wasserstein distance between μ and this *geodesic minimum* is

$$W_p^p(\delta_{\gamma(t_0)}, \mu) = \frac{\lambda(1-\lambda)^{(p-1)p} + (1-\lambda)\lambda^{(p-1)p}}{(\lambda^{p-1} + (1-\lambda)^{p-1})^p} \cdot \operatorname{dist}^p(x, y).$$

Observe that this value is independent from the minimizing geodesic γ of our choice.

Finally, we only have to prove that the minimum can only be achieved inside a minimizing geodesic. For that purpose, we will choose any $a \in X$, and we will construct another point a' inside a minimizing geodesic segment γ verifying $W_p^p(\delta_a, \mu) \geq W_p^p(\delta_{a'}, \mu)$.

The case $\operatorname{dist}(a, y) \geq \operatorname{dist}(x, y)$ is straightforward, as choosing $a' = x$ we have

$$\begin{aligned} W_p^p(\delta_a, \mu) &= \lambda \operatorname{dist}(a, x)^p + (1-\lambda) \operatorname{dist}(a, y)^p \\ &\geq (1-\lambda) \operatorname{dist}(a, y)^p \\ &\geq (1-\lambda) \operatorname{dist}(x, y)^p = W_p^p(\delta_x, \mu). \end{aligned}$$

Now, if $\operatorname{dist}(a, y) < \operatorname{dist}(x, y)$, we can pick a' inside γ at distance $\operatorname{dist}(a, y)$ to y. Observe that $\operatorname{dist}(a, x) \geq \operatorname{dist}(a', x)$ or γ would not be minimizing. Then

$$\begin{aligned} W_p^p(\delta_a, \mu) &= \lambda \operatorname{dist}(a, x)^p + (1-\lambda) \operatorname{dist}(a, y)^p \\ &= \lambda \operatorname{dist}(a, x)^p + (1-\lambda) \operatorname{dist}(a', y)^p \\ &\geq \lambda \operatorname{dist}(a', x)^p + (1-\lambda) \operatorname{dist}(a', y)^p = W_p^p(\delta_{a'}, \mu). \end{aligned}$$

Therefore, the minimum can only be achieved inside minimizing geodesics between x and y and our proof is complete. $\square$

Now, we will apply the preceding proposition to construct measures with multiple projections close to any point in X. We will use this to derive sufficient conditions for attaining $\operatorname{reach}(p, X) = 0$ for all $p \in X$.

Theorem 3.3 ([24])**.** *Let X be a geodesic metric space, and $x \in X$ a point such that there exists another $y \in X$ with the property that there exist at least two different minimizing geodesics from x to y. Then, for every $p > 1$,*

$$\mathrm{reach}(x, X \subset W_p(X)) = 0.$$

In particular, if there exists a point $x \in X$ satisfying that property,

$$\mathrm{reach}(X \subset W_p(X)) = 0$$

for every $p > 1$.

Proof. The probability measure $\mu_\lambda = \lambda\delta_x + (1-\lambda)\delta_y$ will have at least two different points minimizing its distance to X by proposition 3.2.

Now simply observe that $W_p^p(\mu_\lambda, \delta_x) = (1-\lambda)\,\mathrm{dist}(x, y)^p$, which decreases to 0 when $\lambda \to 1$. Hence $\mathrm{reach}(x, X) = 0$ for every $x \in X$ satisfying that property, and therefore $\mathrm{reach}(X \subset W_p(X)) = 0$. □

When X is a Riemannian manifold, some common hypotheses will grant us reach 0. For example, a classic result by Berger (see for example [22, Chapter 13, Lemma 4.1]) proves that our theorem can be applied when X is compact. In this case, for any $p \in X$, there always exists another $q \in X$ such that there exist two minimizing geodesics starting at p to q. More precisely, for every $p \in X$ we can choose a maximum q of the function $\mathrm{dist}(p, \cdot)$ and there will be at least two minimal geodesics from p to q. There is a similar result in [35], where it is shown that for every p, there exists $q \in X$ such that p and q are joined by several minimizing geodesics.

Corollary 3.3.1 ([24])**.** *If M is a compact Riemannian manifold, then*

$$\mathrm{reach}(x, M \subset W_p(M)) = 0$$

for every $p > 1$ and $x \in M$.

Also, we can apply our Theorem 3.3 to the non simply connected case:

Corollary 3.3.2 ([24])**.** *If M is a complete Riemannian manifolds with non–trivial fundamental group (i.e. not simply connected), then*

$$\mathrm{reach}(x, M \subset W_p(M)) = 0$$

for every $p > 1$ and $x \in M$.

Proof. Consider the universal cover $\pi \colon \tilde{M} \to M$. Let $x \in M$, and let $\tilde{x}$ be a point with $\pi(\tilde{x}) = x$. Denote by G the fundamental group of M. We know that G acts on $\tilde{M}$ by isometries and that $G\tilde{x}$ is a discrete, locally finite set. Then, we may take $\tilde{x}' \in G\tilde{x}$ at minimal distance from $\tilde{x}$.

Then we can take a minimizing geodesic $\tilde{\gamma} : [0, \ell] \to \tilde{M}$ from $\tilde{x}$ to $\tilde{x}'$, and the projection $\gamma = \pi \circ \tilde{\gamma}$ will be a geodesic loop such that $\gamma(0) = \gamma(\ell) = x$, and γ is globally minimizing on $[0, \ell/2]$ and $[\ell/2, \ell]$. Otherwise, by taking a shorter curve to the midpoint $\gamma(\ell/2)$ and lifting it we could construct a shorter geodesic from $\tilde{x}$ to another point in $G\tilde{x}$ and our two points would not be at minimal distance. □

Both corollaries can be generalized as follows:

Corollary 3.3.3 ([24])**.** *If a proper geodesic space X is not contractible, then*

$$\mathrm{reach}(x, X \subset W_p(X)) = 0$$

for every $p > 1$ and $x \in X$.

3.2.2 Infinite reach

In this subsection, we are going to use definitions of Section 1.3 and results of [57] that we recall here for the readability of the following proofs:

Lemma 3.1 (Lemma 1.4, [57]). *A uniformly p–convex metric space (X, dist) is uniformly q–convex for all $q \geq p$.*

Theorem 3.4 (Theorem 4.4, [57]). *On any p–convex, reflexive metric space (X, dist) every measure p–moment has p–barycenter.*

Corollary 3.4.1 (Corollary 4.5, [57]). *Let (X, dist) be a metric space with the same hypotheses as in Theorem 3.4. In addition, let $p \in [1, \infty]$ and (X, dist) be strictly p–convex if $p \in [1, \infty)$ and uniformly ∞–convex if $p = \infty$. Then p–barycenters are unique for $p > 1$. In case $p = 1$, all measures admitting 1–barycenters which are not supported on a single geodesic have a unique 1–barycenter.*

Now, we present our results about infinite reach of the canonical isometric embedding into the Wasserstein space:

Theorem 3.5 ([24]). *Let (X, dist) be a reflexive metric space. Then the following assertions hold:*

1. *If X is strictly p–convex for $p \in [1, \infty)$ or uniformly ∞–convex if $p = \infty$, then*

$$\mathrm{reach}(X \subset W_r(X)) = \infty, \textit{ for } r > 1. \tag{3.6}$$

2. *If X is Busemann, strictly p–convex for some $p \in [1, \infty]$ and uniformly q–convex for some $q \in [1, \infty]$, then*

$$\mathrm{reach}(X \subset W_r(X)) = \infty, \textit{ for } r > 1. \tag{3.7}$$

Proof. In Theorem 3.4, Kell establishes that any p-convex and reflexive metric space possesses p-barycenters, as he defined them in [57, Definition 4.3.]. His Theorem 4.4 establishes the existence of such barycenters but not uniqueness. To establish it, we require the conditions we stated in both cases of our theorem. Now we present how these restrictions give us infinite reach.

1. Following Corollary 3.4.1, the spaces (X, dist) which satisfy the hypotheses in item (1) of the theorem have unique r–barycenters for $r > 1$. In other words, every $\mu \in W_r(X)$ has a unique barycenter. This finishes the proof of the first assertion of the theorem.

2. Following Lemma 3.1, if (X, dist) is strictly (resp. uniformly) p–convex for some p, then it is strictly (resp. uniformly) p–convex for all p. Hence, we are in case (1). □

As we pointed out in Section 1.2.2, $\mathrm{CAT}(0)$–spaces are a well-known example of metric spaces satisfying some of the hypotheses in Theorem 3.5. In that sense, there is a straightforward corollary to our Theorem 3.5 in terms of $\mathrm{CAT}(0)$–spaces:

Corollary 3.5.1. *Let* (X, dist) *be a reflexive* $\mathrm{CAT}(0)$*-space, then*

$$\mathrm{reach}(X \subset W_p(X)) = \infty, \textit{ for } p > 1.$$

Proof. As Kell stated in [57], $\mathrm{CAT}(0)$-spaces are both Busemann spaces and uniformly p–convex for every $p \in [1, \infty]$. Let (X, dist) be a reflexive $\mathrm{CAT}(0)$-space and x, y, $z \in X$. From the definition of $\mathrm{CAT}(0)$-spaces, we have that

$$\begin{aligned} \mathrm{dist}(m(x,y),z) &\leq \mathrm{dist}_{\mathbb{E}^n}(m(x',y'),z') \\ &\frac{1}{2}\left(\mathrm{dist}_{\mathbb{E}^n}(x',z') + \mathrm{dist}_{\mathbb{E}^n}(y',z')\right) = \frac{1}{2}\left(\mathrm{dist}(x,z) + \mathrm{dist}(y,z)\right), \end{aligned}$$

where $m(x, z)$ denotes de midpoint between x and z.

Hence, $\mathrm{CAT}(0)$–spaces are strictly 1–convex and, by [57, Lemma 1.4] they are strictly p–convex for all p. The conclusion now follows from item (2) in Theorem 3.5. □

As particular cases of $\mathrm{CAT}(0)$-spaces, we have Hadamard manifolds (complete, simply connected Riemannian manifolds with non-positive sectional curvature everywhere) and, in particular, Euclidean n–space. So, as a corollary, we obtain the following:

Corollary 3.5.2 ([24]). *Let* (M^n, g) *be a Hadamard manifold. Then*

$$\mathrm{reach}(M^n \subset W_p(M^n)) = \infty, \textit{ for } p > 1.$$

In particular,

$$\mathrm{reach}(\mathbb{E}^n \subset W_p(\mathbb{E}^n)) = \infty, \textit{ for } p > 1,$$

where $\mathbb{E}^n$ *is the Euclidean* n*–space.*

In other words, let (M^n, g) *be a Hadamard manifold, then any measure* $\mu \in W_p(M)$ *has a unique* p*–barycenter for* $p > 1$.

Other authors have considered the existence of barycenters in the $\mathrm{CAT}(\kappa)$–space context, specifically $\kappa = 0$. In [83, Proposition 4.3.], Sturm proved the existence and uniqueness of barycenters for $\mathrm{CAT}(0)$–spaces only for the 2–Wasserstein space. In [95, Theorem B], Yokota stated a condition on $\mathrm{CAT}(\kappa)$–spaces, with $\kappa > 0$, to have unique barycenters. This condition is related to the size of the diameter of the $\mathrm{CAT}(\kappa)$–space, which needs to be small in order to have unique barycenters.

Regarding this study of the reach into the Wasserstein space, we have some open questions we state here:

Question 3. Is there any space X whose $\mathrm{reach}(X \subset W_p(X)) > 0$ when $p > 1$?

Question 4. Let (M^n, g) be a manifold without conjugate points, what is the value of

$$\mathrm{reach}(M \subset W_p(M))$$

for $p > 1$?

3.2.3 Projection map

As we have pointed out several times, the reach has a close relation to the metric projection of points of the space onto the image of the embedding. We have studied the regularity of those projection maps in the case of the Wasserstein space.

We define the *projection map* as

$$\begin{aligned} \mathrm{proj}_p : W_p(X) &\to X \\ \mu &\mapsto r_\mu, \end{aligned}$$

that sends each measure to its $p-$barycenter (i.e. the barycenter on the p–Wasserstein space). Recall that this map is defined in the whole $W_p(X)$ if $\mathrm{reach}(, xX \subset W_p(X)) = \infty$, for all $x \in X$, i. e. each measure μ has a unique p–barycenter.

We briefly recall Kuwae's property **B**, (see Section 4.3 in [57] and references therein). Take two geodesics γ, η such that they intersect at an unique point p_0. Assume that for all points $z \in \gamma[0,1]$ the minimum of the map $t \mapsto \|z - \eta_t\|$ is achieved only by the point p_0. Then for every point $w \in \eta[0,1]$ the minimum of the map $t \mapsto \|w - \gamma_t\|$ is achieved only by p_0.

Theorem 3.6 ([24]). *Let $(X, \|\cdot\|)$ be a reflexive Banach space equipped with a strictly convex norm and satisfying property **B**. Then $proj_2$ is a submetry.*

Proof. First let us make a couple observations. From the strict convexity of the norm it follows that between any two points $x, y \in X$ there is a unique geodesic joining them, more precisely, the curve $[0,1] \ni t \mapsto (1-t)x + ty$. In particular this tells us that $m(x,y) = \frac{1}{2}x + \frac{1}{2}y$.

Let $p > 1$ and $x, y, z \in X$. Due to that for every two real numbers $a, b \in \mathbb{R}$ we have that

$$|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p),$$

we obtain that

$$\begin{aligned} \|m(x,y) - z\|^p &= \|\frac{1}{2}x + \frac{1}{2}y - z\|^p \\ &< 2^{p-1}\left(\|\frac{1}{2}(x-z)\|^p + \|\frac{1}{2}(y-z)\|^p\right) \\ &= 2^{p-1}\left(\frac{1}{2^p}\|x-z\|^p + \frac{1}{2^p}\|y-z\|^p\right) \\ &= \frac{1}{2}\|x-z\|^p + \frac{1}{2}\|y-z\|^p \end{aligned}$$

Hence $(X, \|\cdot\|)$ is strictly $p-$convex and so it satisfies the conditions of Theorem 3.5, where this barycenters exist and are unique. Therefore the projection map proj_2 is well defined.

Now notice that

$$\begin{aligned} \|m(x,y) - m(y,z)\| &= \|\frac{1}{2}(x+z) - \frac{1}{2}(y+z)\| \\ &= \frac{1}{2}\|x-y\|, \end{aligned}$$

which implies that for $p > 1$

$$\|m(x,y) - m(y,z)\|^p < \frac{1}{2}\|x - y\|^p,$$

i.e., it is $p-$Busemann. Then the $2-$Jensen inequality (see Section 4.3 in [57]) holds and so in addition we have that by Proposition 4.8 in [57] proj_2 is $1-$Lipschitz. Let $B_r(\mu)$ be a ball in the Wasserstein space. We just proved that

$$\text{proj}_2(B_r(\mu)) \subset B_r(\text{proj}_2(\mu)).$$

Then, it suffices to see that every point in $B_r(\text{proj}_2(\mu))$ is the image of a point (the barycenter of a measure) in $B_r(\mu)$. Fix $\mu, r \geq 0$ and let $b \in B_r(\text{proj}_2(\mu))$. Let T be the translation from $\text{proj}_2(\mu)$ to b. Let us show that $T_{\#}\mu$ has b as a barycenter. For any $a \in X$,

$$\begin{aligned} W_2^2(T_{\#}\mu, \delta_{T(a)}) &= \int_X \|x - T(a)\|^2 \, d(T_{\#}\mu)(x) \\ &= \int_X \|T(x) - T(a)\|^2 \, d\mu(x) \\ &= \int_X \|x - a\|^2 \, d\mu(x) = W_2^2(\mu, \delta_a). \end{aligned}$$

Hence, if $a = \text{proj}_2(\mu)$, then a minimizes the distance from X to μ, and then $T(a) = b$ minimizes the distance to $T_{\#}\mu$.

It remains to see that $T_{\#}\mu$ is contained in $B_r(\mu)$. Choosing $(\text{Id}, T)_{\#}\mu$ as a transport plan in $\Pi(\mu, T_{\#}\mu)$,

$$\begin{aligned} W_2^2(\mu, T_{\#}\mu) &= \inf_{\pi \in \Pi(\mu, T_{\#}\mu)} \int_{X \times X} \|x - y\|^2 \, d\pi(x,y) \\ &\leq \int_X \|x - T(x)\|^2 \, d\mu(x) = \|\text{proj}_2(\mu) - b\|^2 < r^2. \end{aligned}$$

Therefore, $T_{\#}\mu \in B_r(\mu)$. $\square$

Examples of spaces satisfying the assumptions of Theorem 3.6 include Hilbert spaces and L^p spaces (see Examples 4.5, and 4.6 in [59]).

3.3 Reach in the Orlicz–Wasserstein space

As in the previous case, we split the study of the reach in the Orlicz–Wasserstein space (Section 1.5.2) into two parts:

3.3.1 Null reach

We start this section with a simple remark.

Remark. Let $\varphi \equiv Id$. Observe that $\psi \circ \text{dist}$ is a distance when ψ is a positive concave function with $\psi(0) = 0$. Then W_ϑ is a 1-Wasserstein distance for the metric space $(X, \psi \circ \text{dist})$. Therefore,

$$\text{reach}(x, X \subset W_\vartheta(X)) = 0$$

whenever $x \in X$ is an accumulation point, by Theorem 3.2.

Now, for general φ holding the isometric embedding restriction (recall Section 1.5.2), we can replicate Proposition 3.2 for the case where X is isometrically embedded into an Orlicz–Wasserstein space using a more delicate argument.

Proposition 3.3 ([24]). *Let X be a geodesic metric space, and let $x, y \in X$ be two points with $x \neq y$. Consider the probability measure*

$$\mu = \lambda \delta_x + (1 - \lambda)\delta_y,$$

for $0 < \lambda < 1$. Then, the following assertions hold:

1. *μ can only minimize its ϑ–Wasserstein distance to X inside a minimizing geodesic between x and y (i. e. the minimizer lies inside a minimum).*

2. *If λ is close to one, and there exists a constant $c > 1$ such that $\varphi^{-1}(t) < t$ for every $t > c$, then the minimum will be attained inside the interior of each geodesic.*

Proof. First we will see that the minimum can only be attained inside a geodesic. For that purpose, we will replicate the argument in the proof of Proposition 3.2. That is, given $a \in X$, we construct $a' \in \gamma([0, \ell])$, where γ is a minimizing geodesic, with

$$W_\vartheta(\delta_a, \mu) > W_\vartheta(\delta_{a'}, \mu).$$

Again, it suffices to consider the case $\operatorname{dist}(a, y) \leq \operatorname{dist}(x, y)$. We can pick $a' \in \gamma([0, \ell])$ such that $\operatorname{dist}(a', y) = \operatorname{dist}(a, y)$. Then, $\operatorname{dist}(a', x) < \operatorname{dist}(a, x)$ or a is also inside a minimizing geodesic.

Let

$$S = \left\{ t > 0 : \lambda\varphi\left(\frac{1}{t}\operatorname{dist}(a, x)\right) + (1 - \lambda)\varphi\left(\frac{1}{t}\operatorname{dist}(a, y)\right) \leq 1 \right\}.$$

As we have only one transport plan $\pi = \delta_a \otimes \mu$, we can write

$$W_\vartheta(\delta_a, \mu) = \inf S.$$

Thus, it is enough to see that, if t_0 verifies the inequality inside that infimum for a, then it will verify it for a'. Indeed,

$$\begin{aligned} 1 &\geq \lambda\varphi\left(\frac{1}{t_0}\operatorname{dist}(a, x)\right) + (1 - \lambda)\varphi\left(\frac{1}{t_0}\operatorname{dist}(a, y)\right) \\ &= \lambda\varphi\left(\frac{1}{t_0}\operatorname{dist}(a, x)\right) + (1 - \lambda)\varphi\left(\frac{1}{t_0}\operatorname{dist}(a', y)\right) \\ &> \lambda\varphi\left(\frac{1}{t_0}\operatorname{dist}(a', x)\right) + (1 - \lambda)\varphi\left(\frac{1}{t_0}\operatorname{dist}(a', y)\right). \end{aligned}$$

The last inequality comes from the monotonicity of φ, and the assumption

$$\operatorname{dist}(a', x) < \operatorname{dist}(a, x).$$

Observe that, because the previous inequality is strict, we will have a strict inequality in $W_\vartheta(\delta_a, \mu) > W_\vartheta(\delta_{a'}, \mu)$.

Now we will prove the second part of our proposition. Assuming λ close to 1, and that φ differs from the identity for big enough values, we will see that there are points $a \in \gamma((0,\ell))$ with

$$W_\vartheta(\delta_a, \mu) \leq \min\{W_\vartheta(\delta_x, \mu), W_\vartheta(\delta_y, \mu)\}. \tag{3.8}$$

First, we observe that the right-hand side in inequality 3.8 above is easy to compute. Using that φ^{-1} is an increasing function,

$$\begin{aligned}
W_\vartheta(\delta_x, \mu) &= \inf\left\{t > 0 : (1-\lambda)\varphi\left(\frac{1}{t}\operatorname{dist}(x,y)\right) \leq 1\right\} \\
&= \inf\left\{t > 0 : \varphi\left(\frac{1}{t}\operatorname{dist}(x,y)\right) \leq \frac{1}{1-\lambda}\right\} \\
&= \inf\left\{t > 0 : \frac{1}{t} \leq \frac{\varphi^{-1}\left(\frac{1}{1-\lambda}\right)}{\operatorname{dist}(x,y)}\right\} \\
&= \inf\left\{t > 0 : \frac{\operatorname{dist}(x,y)}{\varphi^{-1}\left(\frac{1}{1-\lambda}\right)} \leq t\right\} \\
&= \frac{\operatorname{dist}(x,y)}{\varphi^{-1}\left(\frac{1}{1-\lambda}\right)}.
\end{aligned}$$

Similarly, $W_\vartheta(\delta_y, \mu) = \dfrac{\operatorname{dist}(x,y)}{\varphi^{-1}(1/\lambda)}$. If we want λ close to one, we can suppose $\lambda > 1-\lambda$. Therefore, $1/(1-\lambda) > 1/\lambda$, and because φ^{-1} is increasing,

$$\varphi^{-1}(1/(1-\lambda)) > \varphi^{-1}(1/\lambda).$$

Thus, we know that

$$t_0 := \min\{W_\vartheta(\delta_x, \mu), W_\vartheta(\delta_y, \mu)\} = \frac{\operatorname{dist}(x,y)}{\varphi^{-1}\left(\frac{1}{1-\lambda}\right)}.$$

Now, we will show that we can find a point inside the geodesic $a = \gamma(s)$, $s \in (0,\ell)$ verifying inequality (3.8). It suffices to see that $t_0 \in S$, because $W_\vartheta(\delta_a, \mu)$ is the infimum of S and by definition will be smaller. First, observe that, by monotonicity of φ^{-1}, the inequality defining S is equivalent to

$$\varphi^{-1}\left(\lambda\varphi\left(\frac{1}{t}\operatorname{dist}(a,x)\right) + (1-\lambda)\varphi\left(\frac{1}{t}\operatorname{dist}(a,y)\right)\right) \leq \varphi^{-1}(1) = 1.$$

By the concavity of φ^{-1}, it is enough to have

$$\lambda\frac{1}{t}\operatorname{dist}(a,x) + (1-\lambda)\frac{1}{t}\operatorname{dist}(a,y) \leq 1.$$

We will evaluate $t = t_0$ and look for a condition on s so the preceding inequality is verified. Observe that $\operatorname{dist}(a,x) = s$, $\operatorname{dist}(a,y) = \ell - s$ and $\operatorname{dist}(x,y) = \ell$. Then

$$
\begin{aligned}
\lambda\frac{1}{t_0}\operatorname{dist}(a,x)+(1-\lambda)\frac{1}{t_0}\operatorname{dist}(a,y)\le 1 \iff{}& \lambda\frac{s}{\ell}\cdot\varphi^{-1}(1/(1-\lambda))\\
&+(1-\lambda)\frac{\ell-s}{\ell}\cdot\varphi^{-1}(1/(1-\lambda))\le 1\\
\iff{}& \frac{\lambda s}{\ell}+\frac{\ell-s}{\ell}-\lambda\cdot\frac{\ell-s}{\ell}\le\frac{1}{\varphi^{-1}(1/(1-\lambda))}\\
\iff{}& s\cdot(2\lambda-1)\le\ell\left(\frac{1}{\varphi^{-1}(1/(1-\lambda))}+1-\lambda\right)\\
\iff{}& s\le\frac{\left(\ell\frac{1}{\varphi^{-1}(1/(1-\lambda))}-(1-\lambda)\right)}{2\lambda-1}.
\end{aligned}
$$

If we show that our bound for s is strictly positive, the minimum will be attained inside the geodesic and we will finish the proof. Choosing λ close enough to 1, we have $(2\lambda-1)>0$ and $1/(1-\lambda)>c$. Therefore, $\varphi^{-1}(1/(1-\lambda))-1/(1-\lambda)<0$ and, because the function $t\mapsto 1/t$ is decreasing,

$$\frac{1}{\varphi^{-1}(1/(1-\lambda))}-(1-\lambda)>0$$

and we have finished our proof. □

An immediate consequence of our proposition is the following theorem, providing us with examples of manifolds with zero reach inside their Orlicz–Wasserstein space:

Theorem 3.7 ([24]). *Let X be a geodesic metric space, and $x\in X$ a point such that there exists another $y\in X$ with the property that there exist at least two different minimizing geodesics from x to y. Suppose X is isometrically embedded into an Orlicz-Wasserstein space $W_\vartheta(X)$. Then, for every φ such that $\varphi(t_0)\neq t_0$ for some $t_0>1$,*

$$\operatorname{reach}(x,X\subset W_\vartheta(X))=0.$$

In particular, if there exists a point $x\in X$ satisfying that property, then

$$\operatorname{reach}(X\subset W_\vartheta(X))=0$$

for every $p>1$. Also, for compact manifolds and non-simply connected manifolds,

$$\operatorname{reach}(x,X\subset W_\vartheta(X))=0$$

for every $x\in X$.

Proof. The proof is identical to the one from Theorem 3.3. It remains to see that $\varphi(t_0)\neq t_0$ implies the condition we ask for in Proposition 3.3. Indeed, the convexity and $\varphi(1)=1$ imply $\varphi(t)>t$ for every $t>t_0$. And, because φ^{-1} is increasing, we also have $t>\varphi^{-1}(t)$ for every $t>t_0$, which is what we need to apply Proposition 3.3. □

3.3.2 Positive Reach

Similarly to the p–Wasserstein case, using results of Kell [57], we can prove that reflexive $\mathrm{CAT}(0)$–spaces inside some Orlicz–Wasserstein spaces have infinite reach.

Theorem 3.8 ([24]). *Let (X, dist) be a reflexive $\mathrm{CAT}(0)$–space. Suppose φ is a convex function that can be expressed as $\varphi(r) = \psi(r^p)$, where ψ is another convex function and $p > 1$. Then*

$$\mathrm{reach}(X \subset W_{\vartheta}(X)) = \infty, \tag{3.9}$$

where $\psi \equiv \mathrm{Id}$ and $\varphi(1) = 1$.

Proof. As we pointed out in the proof of Corollary 3.5.1, $\mathrm{CAT}(0)$–spaces are strictly p-convex. Hence by [57, Lemma A.2.] they are strictly Orlicz φ-convex. Thus, the result is derived directly from [57, Theorem A.4.] which confirms the existence of unique barycenters for every $\mu \in W_{\vartheta}(X)$. □

Remark. All proper metric spaces (i.e., those where every bounded closed set is compact) are reflexive [47, 57]. Derived for the proof of [14, Proposition 3.7.], *symmetric spaces* of non–compact type (i.e. with non-positive sectional curvature and no non-trivial Euclidean factor) and *Euclidean buildings* are proper $\mathrm{CAT}(0)$ spaces and are examples for which Theorem 3.8 holds (for more information, read the survey about $\mathrm{CAT}(0)$–spaces of Caprace [21]).

As in the Wasserstein space, we have the open question of finding any space with positive but not infinite reach.

3.4 Reach of the Persistence Diagram Space

Finally, we studied the reach into the third Wasserstein-type space: the space of persistence diagrams. In [18, Theorem 19], Bubenik and Wagner construct an explicit isometric embedding (see Figure 3.1) of bounded separable metric spaces into $(\mathrm{Dgm}_\infty, w_\infty)$.

$$\begin{aligned} \varphi : (X, \mathrm{dist}) &\to (\mathrm{Dgm}_\infty, w_\infty) \\ x &\mapsto \{(2c(k-1), 2ck + \mathrm{dist}(x, x_k))\}_{k=1}^{\infty}, \end{aligned}$$

where $c > \mathrm{diam}(X) = \sup\{\mathrm{dist}(x, y) \colon x, y \in X\}$ and $\{x_k\}_{k=1}^{\infty}$ is a countable, dense subset of (X, dist). The authors stated that this embedding can be thought of as a shifted version of the Kuratowski embedding.

Theorem 3.9 ([24]). *Let (X, dist) be a separable, bounded metric space and $(\mathrm{Dgm}_\infty, w_\infty)$ the space of persistence diagrams with the bottleneck distance. If $x \in X$ is an accumulation point, then*

$$\mathrm{reach}(x, X \subset \mathrm{Dgm}_\infty) = 0.$$

In particular, if X is not discrete, $\mathrm{reach}(X \subset \mathrm{Dgm}_\infty) = 0$.

Proof. For every two points $x, y \in X$, we can construct a persistence diagram P with at least those two points minimizing the bottleneck distance from the diagram P to the embedded space $\varphi(X)$. That P will be a midpoint between $\varphi(x)$ and $\varphi(y)$, so by choosing y arbitrarily close

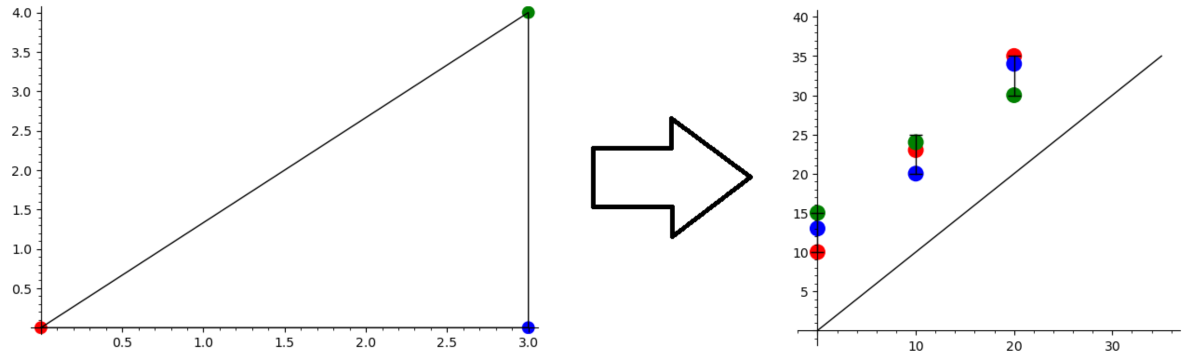

Figure 3.1: Bubenik's embedding of a triangle

to x, we will have a diagram with several barycenters (x and y) that is also arbitrarily close to x. Therefore, $\operatorname{reach}(x, X \subset \mathrm{Dgm}_\infty) = 0$ for every accumulation point $x \in X$, and, thus, $\operatorname{reach}(X \subset \mathrm{Dgm}_\infty) = 0$.

Then, it suffices to prove our first claim. For $x, y \in X$, choose the diagram

$$P = \left\{ \left(2c(k-1), 2ck + \frac{\operatorname{dist}(x, x_k) + \operatorname{dist}(y, x_k)}{2} \right) \right\}_{k=1}^{\infty}.$$

Now, observe that

$$\begin{aligned} w_\infty(\varphi(x), P) &= \sup_{k \in \mathbb{N}} \left| \operatorname{dist}(x, x_k) - \frac{\operatorname{dist}(y, x_k) + \operatorname{dist}(x, x_k)}{2} \right| \\ &= \sup_{k \in \mathbb{N}} \frac{|\operatorname{dist}(x, x_k) - \operatorname{dist}(y, x_k)|}{2} = \frac{1}{2} w_\infty(\varphi(x), \varphi(y)) = \frac{\operatorname{dist}(x, y)}{2}. \end{aligned}$$

And, by a symmetric argument,

$$w_\infty(\varphi(y), P) = \frac{\operatorname{dist}(x, y)}{2}.$$

Note that, similarly to the end of the proof of [18, Theorem 19], any other pairing between points of the diagrams would pair two points from different vertical lines. Those points would be at distance at least $2c$. On the other hand, any possibly unpaired points are at distance at least c from the diagonal. So those pairings would have a cost bigger than $c > \operatorname{dist}(x, y)/2$, and therefore we always pair points in the same vertical lines.

Now, if $z \in X$, we will see that P is at distance at least $\frac{1}{2} \operatorname{dist}(x, y)$ from z. Indeed, we can give a lower bound for the distance simply by omitting the supremum:

$$\begin{aligned} w_\infty(\varphi(z), P) &= \sup_{k \in \mathbb{N}} \left| \operatorname{dist}(z, x_k) - \frac{\operatorname{dist}(x, x_k) + \operatorname{dist}(y, x_k)}{2} \right| \\ &\geq \left| \operatorname{dist}(z, x_k) - \frac{\operatorname{dist}(x, x_k) + \operatorname{dist}(y, x_k)}{2} \right|. \end{aligned}$$

Looking at x_k arbitrarily close to z, we get that

$$w_\infty(\varphi(z), P) \geq \left| \frac{\operatorname{dist}(x, z) + \operatorname{dist}(y, z)}{2} \right| \geq \frac{\operatorname{dist}(x, y)}{2}.$$

This proves that P is not in the image of φ, and that $\varphi(x), \varphi(y)$ both minimize the distance from P to $\varphi(X)$, as we wanted to see. $\square$

Chapter 4

The Rival Coffee Shop Problem

In this final chapter, we present some results concerning the Coffee Shop Problem but with a novel constraint on it: competition. This chapter is based on the paper: Javier Casado and Manuel Cuerno. *The Rival Coffee Shop Problem*. 2023. arXiv: 2304.04535.

Firstly, we present a brief introduction concerning the problem and some of the literature related to our approach.

4.1 The Coffee Shop Problem

Let X be a certain region (e.g., a segment, a square, a torus, or any manifold) and $\{x_i\}$ be a sequence of coffee shops. How can we arrange the coffee shops on X consecutively so that, for any $n \in \mathbb{N}$, the set $\{x_i\}_{i=1}^n$ is placed in the optimal way? This problem, which the reader may know by another name (such as the supermarket or clothing shop problem), is known as the *Coffee Shop Problem*.

Note that this question differs from the problem of finding the optimal arrangement of N shops. To illustrate this, suppose that $X = [0,1] \times [0,1]$ and $N = 4$. If we want to arrange four coffee shops in the optimal way, it seems that dividing X into four squares and placing a coffee shop at the center of each square (see Figure 4.1) would satisfy our condition.

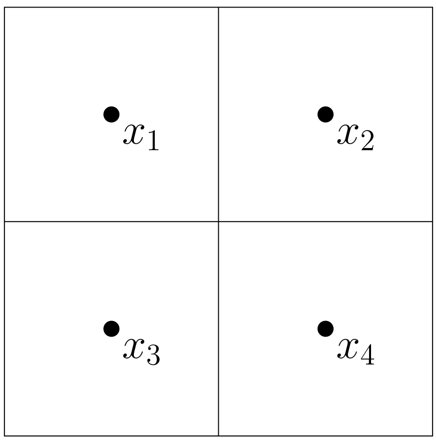

Figure 4.1: Four Coffee shops in the optimal settlement

Whereas, if we follow the original statement of the coffee shop problem, the final configuration for $N = 4$ would be quite different from Figure 4.1. Let $\{x_n\}$ be an infinite sequence of coffee shops and consider the case $N = 4$. To obtain an optimal solution, we must place the first three coffee shops in an optimal way one by one, before placing the fourth one. That is, we first place the first coffee shop in the optimal way, and then, with x_1 fixed, we place x_2 in such a way that x_1 and x_2 are placed in the optimal way, continuing this process until the fourth coffee shop is placed. At each step, the locations of the previous coffee shops (i.e., those with indices $n_0 < N$) are fixed and must be optimal, not just the final configuration at step $N = 4$.

This explanation shows that Figure 4.1 does not correspond to a solution of the coffee shop problem. For example, based on Figure 4.1 but following the coffee shop problem, if we fix $N = 3$, the settlement seems far from optimal (Figure 4.2). Similarly, for $N = 1$, Figure 4.2 shows a suboptimal solution. On the other hand, Figure 4.3 seems to provide a better solution to the coffee shop problem when $N = 4$.

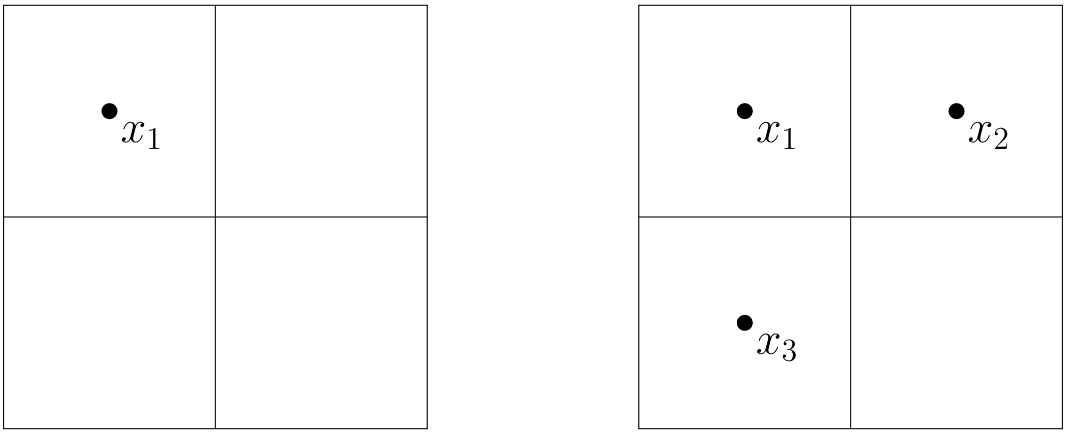

Figure 4.2: On the left: one coffee shop with the setup of Figure 4.1. On the right: three coffee shops with the setup of Figure 4.1

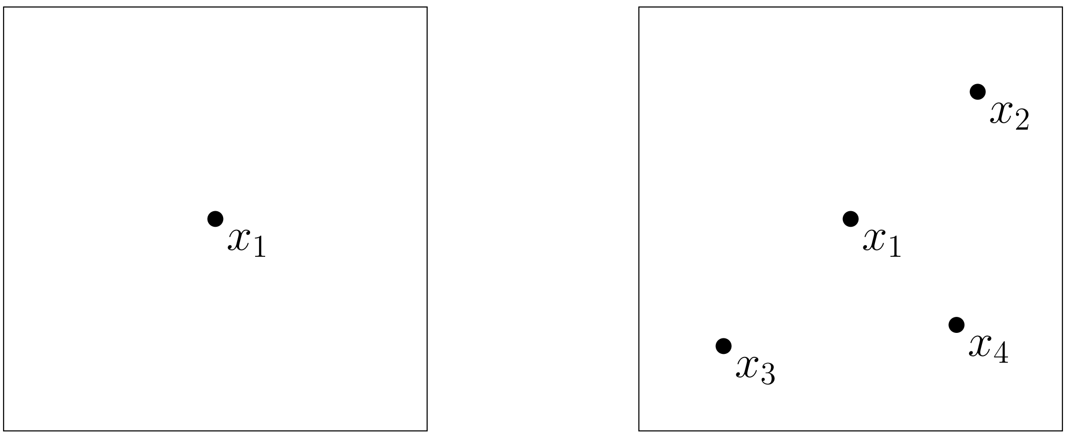

Figure 4.3: On the left: the settlement of the first coffee shop. On the right: a settlement for four coffee shops that seems to fit better the coffee shop problem than Figure 4.1

Let $\{x_i\} \subset X$ be a sequence of points, and suppose we wish to study the optimality of its position if we cut the sequence at step N. This is where the coffee comes in: in real life, if we plan to open an indefinite number of coffee shops, we will not open all of them simultaneously.

Rather, we would start by opening one in the optimal location (such as in the center of a square, as shown in Figure 4.3). If the business is successful, we would then open a second shop, and so on.

4.1.1 State of the art

The Coffee Shop Problem belongs to a extended tradition of localization and optimization problems. As we have pointed at the beginning of the introduction, it can be rephrased with other stores or even different statements, e. g. some optimization problem related to some amount of commodity and warehouses.

Many of these problems can be understood as a way to approximate a uniform distribution by a finite discrete set of points. A theoretical approach to this question is the one developed by the *geometric discrepancy theory* and the interested reader can find more about that perspective in this survey [11]. We also present here more references related this interesting research field [12, 73, 84, 85, 86].

Optimal transport has shown its power to solve many different problems in a vast number of applied scenarios [80, 92]. The Coffee Shop Problem as well as other location, optimization and transportation problems has been also understood in the Wasserstein space context as it is shown by Brown and Steinerberger in [15, 16, 81]. Finally, we want to highlight that, from the probability theory perspective, some work has also been developed and we show here some references in order to seek more information [5, 13, 87].

Following Steinerberger and Brown's work, they understood the Coffee Shop Problem as a way to minimize the following distance:

$$W_2\left(\sum_{i=1}^{N}\frac{1}{N}\delta_{x_i}, dx\right), \tag{4.1}$$

where W_2 denotes the 2–Wasserstein distance, dx is the Riemannian volume measure of our space, normalized with $dx(X) = 1$, $\{x_i\} \subset X$ is a subset and δ_{x_i} denotes the Dirac measure at x_i and represents each Coffee Shop.

In terms of this 2–Wasserstein setting, the Coffe Shop Problem can be reformulated in terms of the following: let $\{x_1, \ldots, x_{N-1}\} \in X$, lets find $x_N \in X$ such that

$$W_2\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, dx\right) = \min_{x_N\in X} W_2\left(\frac{1}{N}\sum_{i=1}^{N-1}\delta_{x_i} + \frac{1}{N}\delta_{X_N}, dx\right).$$

In [81], Steinerberger uses the heat kernel and the Green function to obtain the following result, which will be crucial for the rest of our chapter. Note that he does not fix any particular sequence.

Theorem 4.1 (Steinerberger, [81, Theorem 1]). *Let X be a smooth, compact d–dimensional manifold without boundary, $d \geq 3$, and let $G : X \times X \to \mathbb{R} \cup \{\infty\}$ denote the Green function of the Laplacian normalized to have average value 0 over the manifold. Then, for any set of N points $\{x_1, ..., x_N\} \subset X$, we have*

$$W_2\left(\sum_{i=1}^{N}\frac{1}{N}\delta_{x_i}, dx\right) \lesssim_X \frac{1}{N^{1/d}} + \frac{1}{N}\left|\sum_{k\neq l} G(x_k, x_l)\right|^{1/2}. \tag{4.2}$$

If the manifold is two–dimensional, $d = 2$, we obtain

$$W_2\left(\sum_{i=1}^{N}\frac{1}{N}\delta_{x_i}, dx\right) \lesssim_X \frac{\sqrt{\log N}}{N^{1/2}} + \frac{1}{N}\left|\sum_{k\neq l} G(x_k, x_l)\right|^{1/2}. \tag{4.3}$$

Remark. The notation $\lesssim_X$ has the same meaning as $\leq$, but with a constant on the right-hand side that depends only on the manifold X. This constant is partly explained by Steinerberger in [81, Section 3], but the reader can also consult [6] and [61].

After Theorem 4.1, in [16] Brown and Steinerberger presented the following theorems:

Theorem 4.2 (Brown & Steinerberger, [16, Theorem 1])**.** *Let the even function $f\colon \mathbb{T} \to \mathbb{R}$ satisfy $\widehat{f}(k) \geq c|k|^{-2}$ for some fixed constant $c > 0$ and all $k \neq 0$. Define a sequence via*

$$x_n = \arg\min_x \sum_{k=1}^{n-1} f(x - x_k).$$

Then this sequence satisfies

$$W_2\left(\frac{1}{n}\sum_{k=1}^{n}\delta_k, dx\right) \lesssim \frac{1}{n^{1/2}},$$

where the implicit constant depends only on the initial set, $f(0)$ and c.

Theorem 4.3 (Brown & Steinerberger, [16, Theorem 3])**.** *Let x_n be a sequence obtained as in (4.5) on an d–dimensional compact manifold. Then*

$$W_2\left(\frac{1}{n}\sum_{k=1}^{n}\delta_{x_k}, dx\right) \lesssim_M \begin{cases} n^{-1/2}\sqrt{\log n}, \text{ if } d = 2 \\ n^{-1/d}, \text{ if } d \geq 3. \end{cases}$$

In the preceding theorems, they eliminate the Green term in (4.2) by choosing a *greedy sequence*, which optimally places each coffee shop at every iteration. We have explained in detail this suppression in Section 4.2. In [15], they obtain similar bounds using other sequences, but this time on the torus.

Theorem 4.4 (Brown & Steinerberger, [15, Theorem 5])**.** *Let $d \geq 2$ and let $\alpha \in \mathbb{R}^d$ be badly approximable (recall that $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is badly approximable if for all positive integers $q \neq 0$, we have that $\max_{1\leq j\leq d} ||\alpha_j q|| \geq c(\alpha)/q^{1/d}$, where $||\cdot||$ is the distance to the nearest integer). Then, the Kronecker sequence (perfectly explained in [15, Section 2.2]) satisfies on the torus*

$$W_2\left(\frac{1}{n}\sum_{k=1}^{n}\delta_{x_k}, dx\right) \lesssim_{c(\alpha),d} n^{1/d}.$$

In comparison to the original problem, we introduce competition in the region X. This modification seems natural, as in a city, different coffee brands compete for control over certain areas.

4.1.2 Competition

The key idea for this new setting is to compare a new measure $\mu = \sum_{i=1}^{N_1} \delta_{x_i} - \sum_{j=1}^{N_2} \delta_{y_j}$ with dx, so our shops are represented by the positive deltas, and rival shops by the negative ones. Unlike the problem studied by Steinerberger and Brown, μ is not a probability measure, and in fact, it is not even positive. We have chosen the *Signed Wasserstein distance* $\mathbf{W}_1^{1,1}$ introduced by Piccoli, Rossi, and Tournus in [78] to deal with this new problem. This distance is less restrictive than the canonical Wasserstein and allows us to work with signed finite measures like μ. Due to the monotonicity of the 1–Wasserstein distance (as the signed Wasserstein uses it), we can use Steinerberger and Brown's results without extra difficulty. Using the signed Wasserstein distance, we can now compare μ with dx and obtain bounds on their distance, providing a good setting for our competition problem.

In order to clarify this distance choice we present here two interpretations of the problem we are studying. It seems that the presence of competition would have to negatively affect our brand of Coffee Shops, so to the factor $\sum_{i=1}^{N_1} \delta_{x_i}$ representing our stores we subtract the rivals $\sum_{j=1}^{N_2} \delta_{y_j}$. Indeed, we can interpret (4.1) as a way to measure how close is the benefit of our Coffee Shop placed in $\{x_i\}$ against a benefit produced by placing uniformly stores all around the region X. In that sense, when we introduce competition in X, we need to subtract to our benefit the one produced by the rival. Then, it naturally appears a signed measure μ, so the use of the Signed Wasserstein distance seems accurate to the problem we are dealing with.

Moreover, because $\mathbf{W}_1^{1,1}$ is invariant by translations (see [78, Lemma 19]), we have the reformulation

$$\mathbf{W}_1^{1,1}\left(\sum_{i=1}^{N_1} \delta_{x_i} - \sum_{j=1}^{N_2} \delta_{y_j}, dx\right) = \mathbf{W}_1^{1,1}\left(\sum_{i=1}^{N_1} \delta_{x_i}, dx + \sum_{j=1}^{N_2} \delta_{y_j}\right).$$

Thus our problem is equivalent to the transport of a uniformly distributed population plus a population localized in the rival shops (that is, dx plus $\sum_{j=1}^{N_2} \delta_{y_j}$) to ours $\sum_{i=1}^{N_1} \delta_{x_i}$.

We want to emphasize that in order to make it more comparable to dx, we have decided to normalize μ in a certain sense. Specifically, we redefine it as

$$\mu = \frac{1}{N_1 + N_2}\left[\sum_{i=1}^{N_1} \delta_{x_i} - \sum_{j=1}^{N_2} \delta_{y_j}\right]. \tag{4.4}$$

A discussion on the choice of the constant has been included in Section 4.4. However, for the sake of brevity, we note that the constant $\frac{1}{N_1+N_2}$ is the one that best approximates the real situation we are considering.

In order to develop the new problem (which we will refer to as the "Rival Coffee Shop Problem"), we have considered two different scenarios: fixed and dynamic competition.

4.2 Fixed competition

As we have pointed out in Section 4.1, we divide our study into two cases. In the first one, the competition only opens a fixed number $N_2 > 0$ of Coffee Shops. With the distance described in Section 1.5.1, we want to see how

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N_1 + N_2}\left[\sum_{i=1}^{N_1} \delta_{x_i} - \sum_{j=1}^{N_2} \delta_{y_j}\right], dx\right)$$

behaves for all N_1.

Intuitively, when N_1 is much bigger than N_2, the rival's influence will be very small. We can formalize that:

Theorem 4.5 ([23]). *Let X be a smooth, compact d–dimensional manifold without boundary, $d \geq 3$, and let $G : X \times X \to \mathbb{R} \cup \{\infty\}$ denote the Green's function of the Laplacian normalized to have average value 0 over the manifold and $N_1, N_2 > 0$. Then, for any distinct sets of points $\{x_1, \ldots, x_{N_1}\}$ and $\{y_1, \ldots, y_{N_2}\}$, we obtain*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) \lesssim_{X,N_2} \frac{1}{(N_1+N_2)^{1/d}} + \frac{1}{N_1+N_2}\left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2},$$

where $\mathbf{W}_1^{1,1}$ is the signed Wasserstein distance defined in [67] and in Section 1.5.1 and $z_i = x_i$ from $i = 1$ to N_1 and $z_i = y_{i-N_1}$ for $i = N_1+1$ to N_1+N_2.

Proof. For the sake of simplicity, we denote

$$A = \frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right] \text{ and } B = \frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} + \sum_{j=1}^{N_2}\delta_{y_j}\right].$$

We can decompose $A = A_+ - A_-$ with $A_+ = \frac{1}{N_1+N_2}\sum_{i=1}^{N_1}\delta_{x_i}$ and $A_- = \frac{1}{N_1+N_2}\sum_{j=1}^{N_2}\delta_{y_j}$. Using the definition in 16 we have that

$$\mathbf{W}_1^{1,1}(A, dx) = W_1^{1,1}(A_+, dx + A_-) = \inf_{\substack{\tilde{\mu},\tilde{\nu}\in\mathcal{M}(X)\\ |\tilde{\mu}|=|\tilde{\nu}|}} \left(|A_+ - \tilde{\mu}| + |dx + A_- - \tilde{\nu}| + W_1(\tilde{\mu}, \tilde{\nu})\right).$$

Now, we choose $\tilde{\mu} = B$ and $\tilde{\nu} = dx$. So, we obtain,

$$\begin{aligned}
\mathbf{W}_1^{1,1}(A, dx) &\leq \left(|A_+ - B| + |dx + A_- - dx| + W_1(B, dx)\right) = \\
&= \left|\frac{1}{N_1+N_2}\sum_{j=1}^{N_2}\delta_{y_j}\right| + \left|\frac{1}{N_1+N_2}\sum_{j=1}^{N_2}\delta_{y_j}\right| + W_1(B, dx) = \\
&= \frac{2N_2}{N_1+N_2} + W_1(B, dx).
\end{aligned}$$

Now we will combine the preceding inequality with an upper bound for $W_1(B, dx)$ given in [81, Theorem 1]:

$$W_1(B, dx) \leq W_2(B, dx) \lesssim_X \frac{1}{(N_1+N_2)^{1/d}} + \frac{1}{N_1+N_2}\left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2}$$

Moreover,

$$\frac{2N_2}{N_1+N_2} + \frac{1}{(N_1+N_2)^{1/d}} \leq \frac{2N_2+1}{(N_1+N_2)^{1/d}},$$

due to $N_1 + N_2 > (N_1 + N_2)^{1/d}$. Putting everything together, we obtain the desired result:

$$\begin{aligned}
\mathbf{W}_1^{1,1}(A, dx) &\leq \frac{2N_2}{N_1+N_2} + W_1(B, dx) \lesssim_X \\
&\lesssim_X \frac{2N_2}{N_1+N_2} + \frac{1}{(N_1+N_2)^{1/d}} + \frac{1}{N_1+N_2}\left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2} \leq \\
&\leq \frac{2N_2+1}{(N_1+N_2)^{1/d}} + \frac{1}{N_1+N_2}\left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2} \leq \\
&\leq \frac{2N_2+1}{(N_1+N_2)^{1/d}} + \frac{2N_2+1}{N_1+N_2}\left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2}
\end{aligned}$$

Hence,

$$\mathbf{W}_1^{1,1}(A, dx) \lesssim_{X,N_2} \frac{1}{(N_1+N_2)^{1/d}} + \left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2} \qquad \square$$

Corollary 4.5.1 (Casado & C., [23]). *The explicit dependence on N_2 is*

$$\mathbf{W}_1^{1,1}(A, dx) \lesssim_X \frac{2N_2}{N_1+N_2} + \frac{1}{(N_1+N_2)^{1/d}} + \frac{1}{N_1+N_2}\left|\sum_{k\neq\ell} G(z_k, z_\ell)\right|^{1/2}.$$

Now suppose the sequence z_n is defined in the following way:

$$z_n = \arg\min_x \sum_{k=1}^{n-1} K(x, x_k) \tag{4.5}$$

We will say that such sequence is a *greedy sequence* or that it is *defined in a greedy manner*. Here $K : X \times X \to \mathbb{R}$ are functions of the form

$$K(x, y) = \sum_{k=1}^{\infty} a_k \frac{\phi_k(x)\phi_k(y)}{\lambda_k},$$

where a_k is assumed to satisfy a two-sided bound $c_1 < a_k < c_2$ for all $k \geq 1$ and ϕ_k are the eigenfunctions of the Laplace operator

$$-\Delta\phi_k = \lambda_k\phi_k.$$

We could assume $a_k = 1$, in which case we obtain Green's function.

Theorem 4.6 ([23]). *Let z_n be a sequence obtained in the previous way on a d–dimensional compact manifold with $d \geq 3$ and let $\{x_1, \dots, x_{N_1}\} \subset \{z_i\}_{i=1}^{N_1+N_2}$ and $\{y_1, \dots, y_{N_2}\} \subset \{z_i\}_{i=1}^{N_1+N_2}$ be such that $x_i \neq y_j$ for all i, j. Then*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) \lesssim_{X,N_2} \frac{1}{(N_1+N_2)^{1/d}}.$$

Proof. We will use the same notation as in the proof of Theorem 4.5. In this case, we are going to use another result from Steinerberger together with Brown [16, Theorem 3], which states that for a sequence z_n constructed in a greedy way, we have that

$$W_2\left(\frac{1}{n}\sum_{k=1}^{n}\delta_{z_k}, dx\right) \lesssim_X \frac{1}{n^{1/d}},$$

for $d \geq 3$. So, in our case,

$$\begin{aligned}\mathbf{W}_1^{1,1}(A, dx) &\leq \frac{2N_2}{N_1+N_2} + W_1(B, dx) \leq \\ &\leq \frac{2N_2}{N_1+N_2} + \frac{1}{(N_1+N_2)^{1/d}},\end{aligned}$$

and we obtain our result

$$\mathbf{W}_1^{1,1}(A, dx) \lesssim_{X,N_2} \frac{1}{(N_1+N_2)^{1/d}}, \text{ for } d \geq 3. \qquad \square$$

Proposition 4.1 ([23])**.** *We have a lower bound that is independent of the sets* $\{x_1, \ldots, x_{N_1}\}$ *and* $\{y_1, \ldots, y_{N_2}\}$*. Indeed,*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) \geq \frac{c}{(N_1+N_2)^{1/d}} - \frac{2N_2}{N_1+N_2},$$

where $c > 0$ *is a constant that depends only on the manifold* X*.*

If N_2 is fixed and $N_1 \to \infty$, this is asymptotically as good as [15, Section 1.2].

Proof. By the triangle inequality, we know that

$$\begin{aligned} W_1\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} + \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) &\leq \\ \leq \mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right], dx\right) & \\ +\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} + \sum_{j=1}^{N_2}\delta_{y_j}\right], \frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right]\right)&. \end{aligned}$$

We know a lower bound for the first term, and also, because $W_1^{1,1}$ is invariant by translations,

$$\begin{aligned}\mathbf{W}_1^{1,1}\left(\frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} + \sum_{j=1}^{N_2}\delta_{y_j}\right], \frac{1}{N_1+N_2}\left[\sum_{i=1}^{N_1}\delta_{x_i} - \sum_{j=1}^{N_2}\delta_{y_j}\right]\right) &= \\ = W_1^{1,1}\left(0, \frac{2}{N_1+N_2}\sum_{j=1}^{N_2}\delta_{y_j}\right) \leq \left|\frac{2}{N_1+N_2}\sum_{j=1}^{N_2}\delta_j\right| = \frac{2N_2}{N_1+N_2}&.\end{aligned}$$

The inequality is obtained by choosing $\tilde{\mu} = \tilde{\nu} = 0$ in the infimum inside of $W_1^{1,1}$. Considering both inequalities and the one that Brown and Steinerberger gave in [15, Section 1.2], we have that

$$\frac{c_d}{(N_1 + N_2)^{1/d}} \leq \mathbf{W}_1^{1,1}\left(\frac{1}{N_1 + N_2}\left[\sum_{i=1}^{N_1} \delta_{x_i} - \sum_{j=1}^{N_2} \delta_{y_j}\right], dx\right) + \frac{2N_2}{N_1 + N_2},$$

which clearly implies our result. $\square$

4.3 Non–fixed competition

In this section, we provide an overview of scenarios where the rival's growth rate is comparable to ours. Although there may be a general framework that captures all such cases, we choose to examine each scenario separately for clarity.

4.3.1 Forbidden areas

Up until this point, we have measured victory solely in terms of the signed Wasserstein distance between the difference of the sums of the Dirac deltas and the uniform distribution. An alternative approach is to compute the distance between each individual set of coffee shops and the uniform distribution, as described in the Steinerberger and Brown papers.

In this section, we consider a scenario where our rival has already opened coffee shops and "colonized" a certain area, such that we are unable to open our own shops within that region of our space X. Consequently, our limit as we approach the dx measure will not encompass this region, whereas our rival's will.

The key to our proof lies in the following proposition:

Proposition 4.2 ([23]). *Suppose $\{x_i\}$ is any sequence in $X \backslash B_r(p)$, where $B_r(p)$ is an open ball of center $p \in X$ and radius $r > 0$. Then,*

$$W_1\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}, dx\right) \geq \frac{r}{2}\operatorname{vol}(B_{r/2}(p)),$$

and in particular $\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$ does not converge to dx.

Proof. Suppose γ is an optimal transport plan from $\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$ to the normalized Lebesgue measure dx. By our hypotheses, there is a mass $\operatorname{vol}(B_{r/2}(p))$ outside $B_r(p)$ that has to travel a bigger distance than $r/2$ to arrive to $B_{r/2}(p)$. We can then bound from below the integral of the definition of the Wasserstein distance by $\frac{r}{2}\operatorname{vol}(B_{r/2}(p))$, the distance times the volume:

$$\begin{aligned}\int_{X\times X} \operatorname{dist}(x,y)\, d\gamma(x,y) &\geq \int_{X\backslash B_r(p)\times B_{r/2}(p)} (r/2)\, d\gamma(x,y)\\ &= \int_{B_{r/2}(p)} (r/2)\, dy = (r/2)\operatorname{vol}(B_{r/2}(p)).\end{aligned}$$

In the first inequality we restrict the domain of the integral, so we can bound the distance from below. Then we just use Fubini's theorem and the fact that γ is a transport plan to obtain the Lebesgue measure dy after integrating with respect to x. $\square$

By using this, we conclude with a straightforward corollary:

Corollary 4.2.1 ([23]). *Suppose x_i follows a greedy sequence, and y_j is any sequence omitting an open ball. Then, there exists an N_0 such that, for every $N \geq N_0$,*

$$W_1\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, dx\right) < W_1\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{y_j}, dx\right).$$

In other words, a smart choice of shops will have better results than any sequence that omits a certain region.

Remark. Notice that in Section 1.5.1, if our rival vanishes a certain area with their finite coffee shops, we will still win because their approximation to dx will be worse than ours. For that reason, it is important that the rival experiences some growth during the competition.

4.3.2 Rival growth in terms of ours

We can express the number of rival coffee shops, N_2, in terms of our own, N_1, by defining a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(N_1) = N_2$. This allows us to summarize many specific cases into a single framework.

In this subsection, we will establish conditions that f must satisfy in order for the rival to defeat us. We divide this part into two subsections. Before presenting these conditions, we prove a technical lemma that will be used throughout the rest of this section.

Lemma 4.1. *Let $\mu, \nu \in \mathcal{M}^s(X)$ be two signed finite measures. Then,*

$$\mathbf{W}_1^{1,1}(\mu, \nu) \geq |\mu(X) - \nu(X)|.$$

Proof. It suffices to check the result for positive measures because

$$|\mu(X) - \nu(X)| = |(\mu^+(X) + \nu^-(X)) - (\nu^+(X) + \mu^-(X))|.$$

Now, for $\mu, \nu \in \mathcal{M}(X)$,

$$\begin{aligned} W_1^{1,1}(\mu,\nu) &= \inf_{|\tilde{\mu}|=|\tilde{\nu}|} \left(|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}| + W_1(\tilde{\mu}, \tilde{\nu})\right) \\ &\geq \inf_{|\tilde{\mu}|=|\tilde{\nu}|} \left(|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|\right) \\ &\geq \inf_{|\tilde{\mu}|=|\tilde{\nu}|} \left(|\mu - \tilde{\mu} - \nu + \tilde{\nu}|\right) \\ &\geq \inf_{\tilde{\mu}(X)=\tilde{\nu}(X)} |(\mu - \tilde{\mu} - \nu + \tilde{\nu})(X)| \\ &= |\mu(X) - \nu(X)|. \end{aligned}$$

□

Case $f(N) \geq f(N-1) + 2$

We present a first result for the dynamic case under the hypothesis of the rival coffee shop complex growing a lot faster than ours. Precisely, we will suppose that $f(N) \geq f(N-1) + 2$. That is, whenever we place a shop, our rivals will place two or more.

Theorem 4.7 ([23]). *Let* $\mu_N = \left(\sum_{i=1}^{N} \delta_{x_i} - \sum_{j=1}^{f(N)} \delta_{y_j}\right)$. *If* $f(N) \geq f(N-1) + 2$, *then, for* N_0 *big enough, the rival shops will win for all* $N \geq N_0$, *i.e.,*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}\mu_N, dx\right) > \mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}(-\mu_N), dx\right). \tag{4.6}$$

Proof. The idea behind the proof is that our rival has a winning strategy. That is, at least he is able to copy the placement of our N-th shop with one of his shops because in every turn he adds at least two shops by hypothesis. So, he can copy ours and settle other shops in the remaining non-occupied space in X. We present a formal computation of this explanation.

Suppose our sequence of shops is given by $x_1, \ldots, x_N$. Then, following this procedure we would have

$$\begin{aligned} y_1 &= x_1, \\ y_{f(1)+1} &= x_2, \\ &\vdots \\ y_{f(N-1)+1} &= x_N, \end{aligned}$$

with every other y_j filling the space in a greedy manner.

We would like to clarify two hidden implications before presenting the final inequalities. Firstly, our hypothesis clearly implies that $F(N) \geq 2N$. The other one is that we will call J the set of indexes of y_j that fills X, that is, the ones that do not copy the sequence x_i. It is a straightforward computation that the cardinality of J is $|J| = f(N) - N$.

Now, we are ready to finish our proof. On the one hand, choosing $\tilde{\mu} = \frac{1}{f(N)-N}\sum_{j\in J}\delta_{y_j}$ and $\tilde{\nu} = dx$ gives us

$$\begin{aligned} \mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}(-\mu), dx\right) &= \inf_{|\tilde{\mu}|=|\tilde{\nu}|}\left(|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}| + W_1(\tilde{\mu}, \tilde{\nu})\right) \\ &\leq \left|\left(\frac{1}{f(N)+N} - \frac{1}{f(N)-N}\right)\sum_{j\in J}\delta_{y_j}\right| \\ &\quad + W_1\left(\frac{1}{f(N)-N}\sum_{j\in J}\delta_{y_j}, dx\right) \\ &\leq \frac{2N}{f(N)+N} + \frac{c}{N^d} \leq \frac{2}{3} + \frac{c}{N^d}, \end{aligned}$$

due to Steinerberger results. On the other hand, using Lemma 4.1,

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}\mu, dx\right) \geq 1 + \frac{f(N)-N}{f(N)+N} = \frac{2f(N)}{N+f(N)} \geq 1.$$

Finally, we observe that for any manifold X of dimension $d \geq 3$ we can choose N_0 such that $\frac{c}{N^d} < \frac{1}{3}$ for all $N \geq N_0$ (we remind that $c > 0$ is a constant depending only on the manifold X). Then, for $N \geq N_0$, we conclude that

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}(-\mu), dx\right) \leq \frac{2}{3} + \frac{c}{N^d} < 1 \leq \mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}\mu, dx\right). \qquad \square$$

Case $f(N) = N + K$

Now, we will suppose $f(N) = N+K$. That is, the rival will set one shop every time we do, but he starts with an advantage. In certain sense, we are growing at the same speed. It seems clear that, for big values of N the two chains of shops will be in a very similar position. For that reason, we will study the situation for fixed N.

Theorem 4.8 ([23]). *Let* $\mu_N = \left(\sum_{i=1}^{N} \delta_{x_i} - \sum_{j=1}^{N+K} \delta_{y_j}\right)$*, and* $N_0 > 0$*. Then, there exist values of* K *such that the rival shops will have a winning strategy for all* $N \leq N_0$*, i.e.,*

$$\mathbf{W}_1^{1,1}\left(\frac{1}{2N+K}\mu_N, dx\right) > \mathbf{W}_1^{1,1}\left(\frac{1}{2N+K}(-\mu_N), dx\right). \tag{4.7}$$

Proof. In the same spirit as in Subsection 4.3.2, the rival has, at least, the strategy of choosing their first K shops in a greedy manner, and then $y_{n+K} = x_n$ for all n. In that case, $\mu_N = -\sum_{j=1}^{K} \delta_{y_j}$.

Now, using lemma 4.1 we can see that

$$\mathbf{W}_1^{1,1}\left(\frac{1}{2N+K}\mu_N, dx\right) \geq 1 + \frac{K}{2N+K}.$$

On the other hand, if we choose $\tilde{\mu} = \frac{1}{K}\sum_{j=1}^{K} \delta_{y_j}$ and $\tilde{\nu} = dx$, we obtain that

$$\begin{aligned}
\mathbf{W}_1^{1,1}\left(\frac{1}{2N+K}(-\mu), dx\right) &= \inf_{|\tilde{\mu}|=|\tilde{\nu}|}\left(|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}| + W_1(\tilde{\mu}, \tilde{\nu})\right) \\
&\leq \left|\left(\frac{1}{2N+K} - \frac{1}{K}\right)\sum_{j\in J}\delta_{y_j}\right| + W_1\left(\frac{1}{K}\sum_{j=1}^{K}\delta_{y_j}, dx\right) \\
&\leq \frac{2N}{2N+K} + \frac{c}{K^d} = 1 - \frac{K}{2N+K} + \frac{c}{K^d}.
\end{aligned}$$

In the last inequality we have applied the result of Brown and Steinerberger [15] to the greedy sequence $y_1, \ldots, y_K$. We recall that c is a positive constant which depends only on the manifold X.

Combining both inequalities, we have shown that whenever

$$\frac{c}{K^d} \leq \frac{2N}{2N+K}, \tag{4.8}$$

our result holds. And, by basic calculus, we know that for a fixed $N_0 > 0$ there exist a number $K_0 > 0$ such that for any $K \geq K_0$ and all $N \leq N_0$, the inequality 4.8 is verified. □

We imposed $f(N) \geq f(N-1) + 2$ for two reasons: on the one hand, this restriction implies $f(N) \geq 2N$. On the other hand, due to this we were able to explicitly define the winning sequence (**??**). Losing that clarity we can make a more general statement:

Corollary 4.8.1. *The result also holds if* $\liminf \frac{f(N)}{N} = \lambda > 1$.

Proof. The assumption $\liminf \frac{f(N)}{N} = \lambda$ means that $f(N)$ will increase in a comparable way to λN, so the rival shops will still be able to copy our locations and establish new ones in an optimal way (possibly at a slower rate that once every turn, if $\lambda < 2$).

The bounds

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}(-\mu), dx\right) \leq \frac{2N}{f(N)+N} + \frac{c}{N^d}$$

and

$$\mathbf{W}_1^{1,1}\left(\frac{1}{N+f(N)}\mu, dx\right) \geq 1 + \frac{f(N)-N}{f(N)+N}$$

from the proof of Theorem 4.7 still applies. Bounding $f(N) \geq \lambda N - \varepsilon$ for N big enough yields the result. □

4.4 Discussion about the constants in the difference of Dirac deltas

During the initial steps of the work of this project, Javier Casado and I considered three different constants to multiply the difference of the Dirac deltas

$$\sum_{i=1}^{N_1} \delta_{x_i} - \sum_{j=1}^{N_2} \delta_{y_j}.$$

In this section, we present our considerations about the matter:

1. $\frac{1}{N_1}$ **in the first term and** $\frac{1}{N_2}$ **in the second term**:

   The main objection to this choice is that it gives different masses to the coffee shops of each team if $N_1 \neq N_2$. From our perspective, this does not capture the essence of our problem, as we consider that all coffee shops (regardless of which team they belong to) have the same power of attraction and, formally, the same weight.

   It would be interesting to consider the problem with different weights. For example, one of the teams could be a big consolidated coffee shop chain while the other team is composed of small ones. For that setting, this constant choice could possibly be appropriate.

2. $\frac{1}{N_1-N_2}$ **multiplying both factors**:

   The virtue of this constant is that it normalizes the measure and turns it into a probability measure. In addition, it gives the same weight to each coffee shop. It seems that the fixed case computations of this chapter hold for this constant. The problem with this choice is the case $N_1 = N_2$, which leads to division by zero. Therefore, this constant is not suitable for our problem.

3. $\frac{1}{N_1+N_2}$ **multiplying both factors**:

   This constant gives the correct weight to each coffee shop regardless of the team they belong to. It appeared when we tried to compute the optimality of all coffee shops against the volume measure regardless of which team they belong to. Moreover, as the denominator is always positive, we can use it for every N_1, $N_2 > 0$ and if we join the masses of the two companies it would result in $N_1 + N_2$, i.e., the total population of stores. The combination of deltas may not be normalized, but that is not a problem after the generalization of the Wasserstein distance.

Conclusiones

A lo largo de esta tesis se ha trabajado con los encajes isométricos de espacios métricos dentro de espacios ambiente conocidos tales como $L^\infty(X)$, con (X, dist) un espacio métrico compacto –en especial se han obtenido resultados para variedades riemannianas cerradas– y espacios de tipo Wasserstein.

El Capítulo 2 está dedicado a $L^\infty(M)$ y el Filling Radius. Se han aportado resultados importantes en términos de cotas a este invariante –Teoremas 2.6, 2.7– y otros en relación al Filling Radius intermedio. Tanto $L^\infty(M)$ como el Filling Radius tienen interesantes preguntas abiertas con las que lidiar y se espera en un futuro volver a ellas con el fin de continuar con el trabajo empezado en esta tesis.

En el Capítulo 3, hemos diseccionado el papel del reach en encajes isométricos de espacios métricos tanto en $L^\infty(M)$ como en espacios de tipo Wasserstein. Aunque los resultados obtenidos han sido meticulosos y han ahondado bastante en sus implicaciones globales, alguna pregunta abierta, como la consecución de reach positivo pero no infinito, puede dar lugar a futuros proyectos.

La tesis termina en el Capítulo 4 con una aplicación del estudio de encajes isométricos gracias al *Problema de las cafeterías* añadiéndole un factor de *competencia*. En las Secciones 4.2 y 4.3 se presentan resultados dependiendo del crecimiento de la competencia, dividiéndolos en interesantes casos de estudio. Queda la cuestión de determinar si todos estos teoremas pueden ser replicados para espacios más generales que variedades riemannianas.

Por tanto, el trabajo aportado en este manuscrito muestra la importancia del estudio de encajes isométricos dentro de la geometría métrica demostrando que da un enfoque bastante fructífero a la hora de resolver problemas abiertos y cuestiones que, de primeras, parecen no guardar relación con dicha técnica.

Conclusions

Throughout this thesis, we have focused on isometric embeddings of metric spaces into well-known ambient spaces such as $L^\infty(X)$, where (X, dist) represents a compact metric space (particularly, results have been obtained for closed Riemannian manifolds) and Wasserstein-type spaces.

Chapter 2 is dedicated to $L^\infty(M)$ and the Filling Radius. Significant results have been provided in terms of bounds on this invariant, as shown in Theorems 2.6 and 2.7, along with other findings related to the intermediate Filling Radius. Both $L^\infty(M)$ and the Filling Radius raise intriguing open questions to tackle, and it is expected that in the future, we will revisit these questions to continue the work initiated in this thesis.

In Chapter 3, we have thoroughly examined the role of reach in isometric embeddings of metric spaces, both in $L^\infty(M)$ and in Wasserstein-type spaces. While the obtained results have been meticulous and have delved deeply into their global implications, some open questions, such as the existence of positive but non-infinite reach, may give rise to future research projects.

The thesis concludes in Chapter 4 with an application of the study of isometric embeddings through the *Coffee Shop Problem* introducing a competitive factor. In Sections 4.2 and 4.3, results are presented depending on the growth of competition, categorizing them into intriguing case studies. The question remains whether all these theorems can be extended to more general spaces beyond Riemannian manifolds.

Therefore, the work presented in this manuscript underscores the importance of studying isometric embeddings within metric geometry, demonstrating that it provides a highly fruitful approach for addressing open problems and questions that may initially appear unrelated to this technique.

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